1 Numbers and symbols

The concept of number is the obvious distinction between beast and man. Thanks to number, the cry becomes song, noises acquire rhythm, the spring is transformed into a dance, force becomes dynamic, and outlines figures.

Joseph Marie de Maistre, French author

Contexts

What is this chapter about?

This chapter explains why numbers are important, both in society and in business. It reminds you about negative numbers and explains how to add, subtract, multiply and divide them and how to use brackets. It explains how to make rough estimates before using a calculator and how to round numbers. It goes on to describe how symbols can be used to represent general relationships between quantities and shows how symbols can be treated in a very similar way to numbers. It introduces the idea of a model and explains what a spreadsheet is.

Why is it useful?

Numbers are used everywhere to describe and measure, to allocate resources and to plan ahead. So the basic numeracy skills in this chapter are an essential skill of modern life. Most professional jobs these days require you to use a spreadsheet and a calculator and in Business, Management and Finance, you will frequently have to manipulate numbers and understand simple mathematical expressions which involve symbols, for instance to represent how budgets are allocated to different departments or calculate an interest payment. And any quantitative subject you study either in education or for professional examinations will require the work in this chapter.

Where does it fit in?

The first part of the chapter describes the basic operations of arithmetic, representing numbers and using a calculator, whereas the second part, using symbols instead of numbers, is really the very beginning of the subject of algebra. Your memories of school algebra might put you off but please bear with us – we show you that symbols can be used in exactly the same way as numbers and we start from the very beginning.

What do I need to know?

We assume only that you know how to add, subtract, multiply and divide positive numbers and zero and that you can cope with numbers like 5.123 in which the fractional part is given using decimal places. Most readers will have covered the work in this part of the book when they were younger. We realise that this may have been a long time ago and that not everyone got on with maths at school, so the explanations are thorough and completely self-contained.

Objectives

After your work on this chapter you should be able to:

- add, subtract, multiply and divide positive, negative and zero numbers;
- combine the operations above using brackets;
- make rough estimates before using a calculator;
- understand how to round to so many decimal places or significant figures and how to use scientific notation;
- use symbols to represent the relationships between quantities;
- understand the difference between constants and variables;
- evaluate an expression using particular values for the variables;
- understand the concept of a model;
- understand what a spreadsheet is;
- perform basic operations to simplify expressions containing symbols.

Imagine a world without numbers. Accurate measurement would be impossible. Physical phenomena like air temperature, medical diagnostics such as blood pressure or economic statistics like the inflation rate and unemployment figures would be impossible to quantify. We would be left saying vaguely that 'prices seem to have gone up', or 'tomorrow will be warmer', but further analysis and comparison would be impossible.

Our society allocates resources – raw materials, labour and property – almost entirely by ascribing monetary values to them. Without numbers your prospective career would not exist – no accounts, no economic models, no sales figures.

So we need numbers. They enable us to describe exactly how our world is now, to allocate resources and to plan into the future.

In management, you will have to represent quantities numerically, calculate them, analyse relationships between them and communicate your findings to clients and colleagues. This will often require you to use symbols to represent the quantities of interest. For instance, a general rule for calculating the amount received at the end of a year, on an investment of £A at an annual interest rate of r% is

$$A \times \left(1 + \frac{r}{100}\right)$$

(Don't worry if you don't understand this now – we will see how it is obtained later.)

In the first four sections of this chapter we will only use *numbers* but in the remaining sections we will use symbols as well. The ways of dealing with symbols are just the same as those for numbers – hardly surprising when you think that the symbols are standing in for numerical values anyway.

1 Positive and negative numbers: adding and subtracting

Here is the first diagnostic 'test box'. Take a minute to try it. If you can answer correctly, with no difficulty whatsoever, you could move directly on to Section 2.

test box 1

Can you solve these?

2 + (–5) =? –3 – (–4) =? 3 – (–2) + (–5) =?

Solutions: –3, 1, 0.

Negative numbers

You are already familiar with numbers, and with decimal places. Numbers like 3, 1000, 10.24, and 3.1427 are all *positive* numbers; that is, they are *greater* than 0. For many applications (for example, temperatures in Celsius below freezing, profit and loss or credits and debits in a bank account) and as a tool for many mathematical skills it is useful to be able to talk about *negative* or *minus* numbers. These are numbers like –5, –7.74, –1000 or –1.0. We will often enclose them in brackets like (–5) or (–7.74) to show that the minus sign belongs to that number.

All these numbers, positive, negative and zero, can be represented on a line. Part of it is shown below. The dots at each end show that the line could continue forever in both directions.

... - + ---- + ---- + ---- + ---- + ---- + ---- + ---- + ---- + ---- + ---- + ---- + ---- + - ...

–6 –5 –4 –3 –2 –1 0 1 2 3 4 5 6

Adding and subtracting negative numbers

We all know that 4 + 5 = 9, or 8 – 3 = 5, but it is less obvious how to add and subtract negative numbers.

Any sum can be represented by a traveller on a journey along the line above. He starts at one point on it and from there can go forwards (to the right) to higher numbers or backwards (to the left) to lower numbers. Adding and subtracting represent forward and backward progress respectively. For instance, for (–5) + 3, the traveller starts at –5 on the line and then moves 3 units forwards to arrive at –2, so (–5) + 3 = (–2). For (–4) – 1 she starts at –4 and moves one step backwards to (–5) so (–4) – 1 = (–5).

check this ▷

Use the line to convince yourself that (–3) + 2 = (–1), that (–3.7) + 1.7 = (–2) and that (–1) – 3 = –4.

When we need to add or subtract a *negative* number the direction must be reversed because the number is negative. For example, to calculate 3 + (–2), he starts at 3 on the line, would move forward for the addition but does the reverse because we are adding a negative number. So he moves 2 units backwards from 3 to arrive at 1 and we conclude that 3 + (–2) = 1. In a similar way, to calculate 3 – (–2) he starts at 3, would move backwards for the subtraction but does the reverse because of the –2 and so moves 2 units forwards to 5.

check this ▷

Convince yourself that (–3) + (–2) = (–5) and that (–1) – (–2) = 1.

Notice that as 3 + (–2) results in moving 2 units backwards it gives exactly the same result as 3 – 2 and that as 3 – (–2) results in moving 2 units forwards it is the same as 3 + 2. This is because 'forwards' and 'backwards' are the reverse of each other. For 3 – (–2) the effect is one of a double negative, the phrase 'I am *not not* going' or the opposite of an opposite.

So, plus a minus number is the same as minus a positive number (opposite signs), whereas minus a negative number is the same as plus a positive number (same signs) as shown below.

Adding and subtracting negative numbers

OPPOSITE SIGNS
+ (– number) or – (+ number) gives a –

SAME SIGNS
– (– number) or + (+ number) gives a +

When evaluating sums it is usually easiest to rewrite them in terms of positive numbers only as we have done in the following examples.

check these ▷

3 + (–5)
= 3 – 5 = –2.

1 – (–4)
= 1 + 4 = 5.

(–3) – 7 = –10.

12.42 – (–3.1)
= 12.42 + 3.1 = 15.52.

When more than two numbers appear in a sum we work in the same way. For instance 3.2 – (–3) + (–2) – 1 = 3.2 + 3 – 2 – 1 = 3.2.

check this ▷

	–6.7	+	(–7)	–	(–0.1)	+	2.1		
=	–6.7	–	7	+	0.1	+	2.1	=	–11.5

2 Positive and negative numbers: multiplying and dividing

test box 2

Can you do these?

(–3) × 2   (–4) × (–0.5)   –2 × 10   8 ÷ (–2)

–6 ÷ (–3)   –4 × 12 × (–2)

Solutions: (row-wise) –6, 2, –20, –4, 2, 96.

Multiplying negative numbers

Multiplication means 'times' so 2×3 is really 'two threes' or $3 + 3$, 4×3 is 'four threes' $3 + 3 + 3 + 3$ and so on. The order of multiplication does not matter so 2×3 is the same as $3 \times 2 = 2 + 2 + 2$, and 4×3 is the same as $3 \times 4 = 4 + 4 + 4$ and so on.

The number line in Section 1 can help us to work out how to multiply negative numbers.

The multiplication 2×3 is $3 + 3$, so our traveller starts at 0 and travels forward 3, and then forward 3 again to reach 6. By similar reasoning $2 \times (-3) = (-3) + (-3)$, so he starts at 0 and then travels *backwards* 3 steps and then backwards 3 steps again to reach –6, so we have that $2 \times (-3) = -6$. If the negative number comes first, we can reverse the order of multiplication. For instance, to calculate $(-4) \times 2$ we regard this as $2 \times (-4)$ and work out $(-4) + (-4) = (-8)$.

The big problem comes when we want to multiply two negative numbers together, let's say $(-5) \times (-2)$. How can our traveller do backward steps of 2, *minus* 5 times? Or backward steps of 5, *minus* 2 times? Here we do need a leap of faith. The convention is that *a negative times a negative is a positive*. This rule was adopted because everything then falls into a pattern for later work.

The key results for multiplying positive and negative numbers are:

Multiplying numbers

of the **same** sign **gives a +**
of a **different** sign **gives a –**

that is,

$+ \times + = +$
$+ \times - = -$
$- \times + = -$
$- \times - = +$

Of course, to multiply more than two numbers together we just multiply the first two, then multiply the result by the third number and so on. A property of multiplication is that we will obtain the same result regardless of the order in which we multiply the numbers. For instance,

$$\begin{aligned} & 2 \times 3 \times 4 \times 5 \\ &= 6 \times 4 \times 5 \\ &= 24 \times 5 = 120 \end{aligned}$$

or we could have said

$$\begin{aligned} & 2 \times 3 \times 4 \times 5 \\ &= 2 \times 12 \times 5 \\ &= 2 \times 60 = 120 \end{aligned}$$

The result of multiplying two or more numbers together is called the *product* of those numbers. For instance 6 is the product of 2 and 3 and 100 is the product of 5 and 20.

check this ▷

What is the product of 6 and 30? Of 2 and –5 and 20?

Answers: 180 and –200.

Dividing negative numbers

The rules for division have to comply with those for multiplication. For example, because 4 × 5 = 20, 20 divided by 4 must be 5 and 20 divided by 5 must be 4. So the rules we have already met for multiplication using negative numbers dictate the rules for division. For instance, as 4 × (–5) = –20, we can deduce that (–20) ÷ (–5) = 4 and that (–20) ÷ 4 = –5.

Dividing numbers

of the **same** sign **gives a +**
of a **different** sign **gives a –**

that is,

+	÷	+	=	+
+	÷	−	=	−
−	÷	+	=	−
−	÷	−	=	+

Notice that the rules for the sign of a division using negative numbers are just the same as the rules for multiplication; that is, dividing numbers with *different* signs gives a negative result, whereas dividing those of the *same* sign gives a positive result.

check these ▷

10 ÷ (–5) = (–2)

–100 ÷ 20 = –5

–28 ÷ (–7) = 4.

Dividing by 0

It is *not* possible to divide by 0. How can a quantity be split into 0 parts?

Alternatives to the ÷ sign

You may remember that division can be written in several ways. The division 8 ÷ 2 can also be written $\frac{8}{2}$ or 8/2 and the result is called the *quotient*. Writing a division in this way is particularly useful when a whole expression needs to be divided by another whole expression. For instance, the division of 80 – 20 by 5 + 10 can be written

$$\frac{80-20}{5+10}$$

Be especially careful about the exact length of the quotient line. For instance

$$1+\frac{5+3}{2}$$

(which equals 5) is crucially different from

$$\frac{1+5+3}{2}$$

(which equals 4.5).

Work card for 1 and 2

1. Evaluate the following:
   a. $6 + (-3) - (-4)$ b. $(-3) + (-7) - 11 - (-8)$
   c. $0 - (-3) - (-1) - (-4)$ d. $-4 + 3 - (-2)$
   e. $-5 - (-2) + (-4)$

2. Evaluate the following:
   a. $6 \times (-3)$ b. -4×5 c. $(-8) \times (-8)$
   d. $6 \times (-1)$ e. $(-5) \times (-10)$ f. $(-60) \times (-2)$
   g. $(-2) \times (-60)$ h. What is the product of 2 and 20?

3. Evaluate the following:
   a. $10 \div (-2)$ b. $60 \div (-15)$ c. $-12 \div 6$ d. $-12 \div (-6)$
   e. $\frac{12}{6}$ f. $\frac{12}{-6}$ g. $\frac{-12}{6}$ h. $\frac{-12}{-6}$

4. Evaluate the following:
   a. $(-4) \times (-3) \times (-5)$
   b. $(-4) \times 3 \times (-5)$
   c. $(-1) \times (-1) \times (-1) \times (-1) \times (-1) \times (-1)$
   d. $(-1) \times (-1) \times (-1) \times (-1) \times (-1) \times (-1) \times (-1)$
   e. $-1 \times -5 \times -10$
   f. $-1 \times -5 \times 10$

Solutions:

1. a. 7 b. –13 c. 8 d. 1 e. –7

2. a. –18 b. –20 c. 64 d. –6 e. 50 f. 120 g. 120 h. 40

3. a. –5 b. –4 c. –2 d. 2 e. 2 f. –2 g. –2 h. 2

4. a. –60 b. 60 c. 1 d. –1 e. –50 f. 50

Assessment for 1 and 2

1. Calculate the following:
   a. $(-20) - (-10\ ) - (-5)$ b. $0 - 20 - (-10) + (-5)$
   c. $(-1) - (-1) - 1 + (-1)$

2. Evaluate:
   a. $10 \times (-5)$ b. $(-2) \times 6$ c. $(-5) \times (-10) \times (-3)$
   d. $\frac{-200}{10}$ e. $\frac{200}{-10}$
   f. $-2 \times -2 \times -2 \times -2 \times -2$
   g. $-2 \times -2 \times -2 \times -2 \times -2 \times -2 \times -2$
   h. $\frac{-60}{-3}$ i. $\frac{-40}{8}$ j. $\frac{40}{-8}$ k. $\frac{-40}{-8}$

3 Combining addition, subtraction, multiplication and division

test box 3

$4 \times 8 - 6 \div 2$ $3 \times (50 - 10)$ $\left(\frac{40-20}{10-5}\right) \times 3$

$((75 \div 25) + 2) \times 5$

Solutions: 29, 120, 12, 25.

The order of operations

Calculate $3 - 2 \times 4$.

It may look simple enough, but there is a snag. There are two ways to proceed and they each give a different answer. You might have reasoned:

(i) subtract first to give $3 - 2 = 1$ and then calculate $1 \times 4 = 4$ so the answer is 4

or you might have said

(ii) multiply first, so $2 \times 4 = 8$, and $3 - 8 = -5$.

Which one did you do?

The answer depends on the order in which the calculations are performed – whether to subtract or multiply first – and so longer expressions may have many more than two alternative answers. This situation is unsatisfactory so we need some sort of rule which tells us in which order to perform the operations.

The accepted rule is to *multiply and divide first*, performing calculations from left to right and *then add and subtract*, also from left to right. So for the example above we should multiply first, so the second answer, $3 - 2 \times 4 = 3 - 8 = -5$, is correct.

Let's try $6 \times 2 \div 4 + 1$. The rule tells us to multiply and divide first, but there is both a multiplication and a division here, so we must work from left to right. The multiplication occurs before the division, so we multiply 6×2 first to give

$$12 \div 4 + 1,$$

then divide giving

$$3 + 1,$$

and finally do the addition,

$$= 4.$$

check these ▷

Remember to perform calculations from left to right if there is more than one multiplication/division or more than one addition/subtraction.

$$\begin{aligned} & -2 + 7 \times 8 \div 2 \times 2 \\ = & -2 + 56 \div 2 \times 2 \\ = & -2 + 28 \times 2 \\ = & -2 + 56 = 54 \end{aligned}$$

	50	–	32	÷	16	×	2	
=	50	–			2	×	2	
=	50	–				4	=	46

Introducing brackets

Suppose we need to multiply 2 + 3 by 4 – 2. We *cannot* write this as

$$2 + 3 \times 4 - 2$$

because applying the order of operations rule would give 2 + 12 – 2 = 12. To show that the numbers must be processed in a different order we use *brackets*.

When part of an expression must be evaluated first it must be enclosed in brackets. So the multiplication of 2 + 3 by 4 – 2 must be written (2 + 3) × (4 – 2). This indicates that we must first calculate 2 + 3, and then calculate 4 – 2, and finally multiply the results 5 × 2 = 10.

As evaluating the expressions in brackets takes priority over anything else, we can extend the rule for the order of operations to *brackets*, multiply and divide, add and subtract.

For example, to evaluate (9 – 2) × 10 – (2 × 3) we work out the brackets first to give

$$7 \times 10 - 6$$

then multiply to give

$$70 - 6$$

and finally subtract to obtain

$$= 64.$$

The order of operations in evaluating arithmetic expressions is

Brackets
Multiply and Divide (from left to right)
Add and Subtract (from left to right)

When multiplying by a bracket it is usual to omit the multiplication sign. So (2 + 3) × (4 – 2) is written (2 + 3)(4 – 2) and 2 × (4 + 7) is written 2(4 + 7).

check these ▷

	6	(4 – 6)	(4 + 2)	+	3	
=	6	×(–2)	×6	+	3	
=		(–12)	×6	+	3	
=			(–72)	+	3	= –69

	36	÷	(4 × 3)	–	5	
=	36	÷	12	–	5	
=			3	–	5	= –2

Quotients can be written using brackets, for instance,

$$\frac{80-20}{10-5}$$

can be written (80 – 20) ÷ (10 – 5) because everything *above* the line of a quotient is divided by everything *below* the line.

Sometimes more than one layer of brackets is necessary. Do not be put off by this. You will find that you have to work out the *inside* brackets first and then proceed outwards. For instance

$$(6(1 + 4)) \div 10 = (6 \times 5) \div 10 = 30 \div 10 = 3.$$

Authors and lecturers are sometimes helpful and use different symbols for different 'layers' of brackets. For instance {[(2 + 3) + 5] × 7}. As you will not always encounter this we have often used only one symbol in our work.

check this ▷

$$\begin{aligned} & 10 \times (2 + (6 \div 3) \times 4) \\ = {} & 10 \times (2 + 2 \times 4) \\ = {} & 10 \times (2 + 8) \\ = {} & 10 \times 10 = 100 \end{aligned}$$

In practice, brackets are often used to clarify expressions when they are not strictly essential.

Work card 3

1. Evaluate the following:

a. (20 – 5) × (4 – 2) **b.** 2 + (10 ÷ 5) × 3
c. 2 × (10 ÷ 5) × 2 **d.** 2 × 10 ÷ (5 × 2)
e. $\frac{10+20}{4-2}$ **f.** 2 × 2 × (27 ÷ 3) + (1 – 20)
g. (4 × 2 × 2) + (5 × (–1))

2. Evaluate the following:

a. 1 + 3 × (4 + (8 ÷ 2)) **b.** ((50 ÷ 25) × 8 ÷ (7 – 3)) × 3
c. 12 – (4 (8 × (6 – 4)) – 5) **d.** $\frac{18\times(2-3\times4)}{(4+14)}$
e. $\left(\frac{18\div3}{(4\times3)-36}\right)\times4$
f. $(-10)\times\left(\left(\frac{100}{25}\times2\right)+(60\div20)\right)+1$
g. $1+\left(\frac{2\times(2+4\times5)}{(2\times11)\div(22\div2)}\right)$

Solutions:

1. a. 30 **b.** 8 **c.** 8 **d.** 2 **e.** 15 **f.** 17 **g.** 11

2. a. 25 **b.** 12 **c.** –47 **d.** –10 **e.** –1 **f.** –109 **g.** 23

Assessment 3

1. Evaluate:

a. (40 ÷ 2) + (3 × 4)

b. –5 + (–3) × 2 + 1

c. 6 ÷ (3 × 2) + 4

d. 3 × 3 × (6 ÷ 2) + 3

e. 3 × 3 × 6 ÷ 2 + 3

f. $\frac{3+6}{5-2}$

g. $\frac{10-2}{3+1} \div (-2)$

2. Evaluate:

a. (21 ÷ 7) + (50 ÷ (5 × 2) + 1)

b. $\frac{77 \div 11}{108 \div (-3) \times 4}$

c. $\left(\frac{40}{6+2}\left(3+\frac{5-(21\div 7)}{6-4}\times\frac{48}{16}\right)\right)+5$

d. $\left(20\,000 / \left((1000-(2\times 5\times 50))\times\left(\frac{100}{50}\right)\right)\right)-2$

e. $\frac{30+3\times 3}{10+3}+5\times\frac{10+100}{11}$

4 Using a calculator

test box 4

1. Use a calculator to evaluate:

   $\frac{500}{1.2(20+34)}$ to 6 decimal places.

2. Express the following to 3 decimal places:

   1.9755 10,002.9999 209.452 12.73 0.000123456

3. Express the numbers in question **2** to 3 significant figures.

4. Express the following in scientific notation:

   12,000,000 0.00001254

5. By estimating roughly, do you think that

   $\frac{515\times 6.1}{200}=7.1$

   is correct?

Solutions:

1. 7.716049 **2.** 1.976 10,003.000 209.452 12.730 0.000

3. 1.98 10,000 209 12.7 0.000123 **4.** 1.2×10^7 1.254×10^{-5}

5. Use rough estimates: 500 × 6 = 3000, divided by 200 gives 15, the answer doesn't look good.

Which sort of calculator?

You will need a fairly basic calculator with the usual +, –, ×, ÷ and also simple functions such as $\frac{1}{x}$, $\sqrt{x}$, x^2, log and ln (natural logarithm), 10^x, e^x, x^y. A memory would also be helpful.

The order of operations on a calculator

Like any other invaluable tool a calculator is only as good as its operator – the modern adage, 'garbage in garbage out' is extremely relevant here. The calculator will only produce the right answer if you supply the numbers and operations in the correct order – which may not be the order in which they are written.

Try to evaluate the following expressions using your calculator:

$$(20 + 30) \div 3$$

$$20 + (30 \div 3)$$

The answers you should have are 16.666666 and 30. To calculate the first expression you need to enter 20 + 30 on your calculator and then divide by 3. For the second one you need to evaluate the bracket 30 ÷ 3 first and *then* add 20. We should add that some calculators do provide bracket functions, which we suggest that you use with care.

Show your working

Think about the errors you make when you word-process, or type or write! These are usually apparent when you read through later. It is just as easy to press the wrong key on a calculator, but most machines won't display all your inputs. It is therefore a good idea to write out your intermediate workings and your train of thought for a problem and not just the final value the calculator gives you. By doing this mistakes are easier to spot, your work is easier to follow – for a colleague or for yourself later – and last but not least, if your answer is wrong but your method is right you will still get most of the available marks in an exam.

Rough estimates

Always keep in mind the *real* problem you are solving.

Be critical of the answer your calculator gives you. Look out for percentage decreases that are over 100, negative probabilities, or fractions when you expected whole numbers.

Other errors may be less obvious and so it is a good idea to perform a rough mental calculation to get an idea of the magnitude of the solution.

To illustrate this, consider the following scenarios – and their solutions. Which ones seem reasonable and which don't? *Don't* use a calculator.

check these ▷

1. A university has 6782 students, about half of which can be expected to visit a campus catering outlet on any given term-time day. Lunches are £2–3 and coffee and a snack is about £1. The total takings for a day over all outlets is £6142. Does this seem reasonable?
2. At a bank there is a single queueing system and five cashiers. On average, during peak hours a customer arrives every 32 seconds. The situation is modelled mathematically (such models are called queueing models) to assess the effects of increasing or reducing the number of cashiers on duty. The final result from the model shows that

on average 76% of customers would have to wait longer than five minutes if there were 4 cashiers, 56% if there were 5 cashiers and 66% if there were 6 cashiers.

3. I earn $8.42 an hour and worked 96½ hours last month. My payslip for the month says $585.19. Does this seem reasonable?
4. The interest I will earn on £6179 invested at a rate of 5% for the first £5000 and 7.2% on the remainder over a year is given by

$$5000 \times \frac{5}{100} + 1179 \times \frac{7.2}{100}$$

Using a calculator I obtain £334.88. Does this seem right?

Solutions:

1. Working in thousands, roughly 4000 students will visit an outlet, and if an equal number have coffee or lunch they will spend an average of about £1.75, making total revenue for the day about £7000. The result is about right.
2. The results are suspicious here as more cashiers should bring down the percentage of people who have to wait longer than 5 minutes, whereas here the figure for 6 cashiers is greater than for 5. Maybe the modelling procedure is inappropriate, or else a calculation is erroneous.
3. Approximately $8 for roughly 100 hours should give me $800, so something is wrong. In fact this wage is for 69.5 hours!
4. Yes, it seems reasonable. Say that the average rate of interest is about 6% and that the sum invested is about £6000. We would expect the interest to be about

$$6000 \times \frac{6}{100} = £360$$

Rounding: decimal places and significant figures

When you use a calculator or computer to perform a calculation the machine will only display a certain number of digits. The exact answer may need many more digits or even an infinite number of them. For instance, when we divide 5 by 17 our calculator shows 0.294117647.

Numbers can be rounded to a particular number of *decimal places* (*d.p.*) or a particular number of *significant figures* (*sig. fig.*).

The convention for rounding to a particular number of decimal places (d.p.) is that when the first digit to be *excluded* is between 5 and 9, we round up, and when it is between 0 and 4 we round down. So, for instance, 3.625 expressed to 2 decimal places rounds up to 3.63, and 3.624999 rounds down to 3.62. Remember to include 0s where appropriate. For instance, 3.634999 to 4 decimal places is 3.6350, and to 5 decimal places is 3.63500.

A second way of representing numbers approximately is to write them to a particular number of *significant figures* (sig. fig.). The left-most digit of a number is the most significant as it is the digit that represents the greatest value, the next from left-most is the second most significant and so on. So in the number 672.34 the '6' represents hundreds and is therefore the most significant figure, the '7' represents the number of tens and is the second most significant figure and so on.

To write a number to, say, 3 significant figures we use the three left-most digits, rounding the final one if necessary. As when rounding to so many decimal places, if the first *discarded* digit is 5 or more we round up and if it is between 0 and 4 we round down. For example, 6248.500052 to 3 sig. fig. is 6250, to 2 sig. fig. is 6200 and to 8 sig. fig. is 6248.5001.

check this ▷

What is 7,254,600 to 3 sig. fig., to 4 sig. fig?
What is 0.00652445 to 3 sig. fig., to 5 sig. fig.?

Solutions: 7,250,000, 7,255,000, 0.00652, 0.0065245.

Notice from the last example that any zeros after the decimal point but before the first non-zero digit do not count as 'significant'.

Most calculators display numbers to 8 or 10 significant figures.

In practice, the accuracy with which we need to record results depends on the application. You do not see newspaper headlines reporting that the inflation rate is 3.42534213%: it is usually given to just one decimal place; that is, 3.4%. A chemist analysing a substance may have measuring equipment that is only accurate to 0.001g so it is pointless to record the result to 6 decimal places. A statistical model that forecasts the percentage dividend payable by a company for the next 10 years to 32 decimal places is itself only an approximation, so it is meaningless to report the forecasts to more than perhaps 1 or 2 decimal places.

So, common sense must prevail when deciding how many decimal places or significant figures to give in the *answer* to a problem. However, it can be dangerous to round too much *during* your calculations. Consider the following example.

Past records show that 8892 2cm tacks were manufactured at a cost of £13.16. A management accountant needs to estimate the price at which the factory should sell a batch of 20,000 tacks and reasons as follows.

Cost of manufacturing 8892 tacks is £13.16.

$$\text{Cost per tack is } \frac{13.16}{8892} = £0.001 \text{ rounded to 3 d.p.}$$

A batch of 20,000 tacks therefore costs 20,000 × 0.001 = £20.

She concludes that if the factory sells a batch for £25 it will make a profit.

This is obviously an extreme example and we hope that you can see the source of error here. The accurate calculations are Cost per tack is

$$\frac{13.16}{8892}$$

so a batch of 20,000 tacks costs

$$20{,}000 \times \frac{13.16}{8892} = £29.60$$

so the factory would make a *loss* if it sold a batch for £25.

Computers and calculators perform all their calculations to a particular accuracy. When you write out calculations by hand you may round your intermediate values to a different number of decimal places and so your results may differ slightly from those of the machine.

Scientific notation

You may have noticed that when your calculator is faced with a very large or a very small number it resorts to another notation called *scientific notation*. For instance, calculate 123,456,789 × 1234 on your machine.

Our calculator display shows

1.523456776 11

The small raised 11 means 'multiplied by 10 to the power of 11'.

We will cover powers more thoroughly in Chapter EM2, but for now it is enough to know that 10 to the power of 11 is written 10^{11} and means 10 multiplied by itself 11 times, or

$$10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10.$$

So the number above can be written $1.523456776 \times 10^{11}$.

To multiply a number by 10 you may recall that we just move the decimal point one place to the right; for example, $1.523456776 \times 10 = 15.23456776$. So to multiply by 10, eleven times, we move the decimal point 11 places to the right. The number represented above is therefore 152345677600.0. Remember that this result is not likely to be exact as the calculator only displays a certain number of significant figures.

Very small numbers can be represented using negative powers of 10. Again, these are described in Chapter EM2. Multiplying by 10^{-5} is the same as dividing by 10^5 or 100,000. We can do this by moving the decimal point 5 places *to the left*. So if the diameter of an atomic particle is 0.000000000000000000014 mm it is much easier to write it as 1.4×10^{-20}, where multiplying a number by 10^{-20} equates to moving the decimal point 20 places to the left.

Work card 4

1. Use a calculator to evaluate the following:

$$\frac{24.15}{5(150+11)} \qquad \frac{42-20.04}{366 \times 24} \qquad \frac{120 \div 15}{85+35}$$

2. *Without* using a calculator, decide which of the following answers are most likely:

a. $\frac{312.42 \times 7.54}{21}$ = 112.17366 or 1.1217 or 0.01122

b. 112 articles are purchased at £5.42 and a further 62 articles at £2.42. The total bill is £757.08, £1050.12 or £342.01?

c. I purchase 300 euros at an exchange rate of 1.5 euro to £1. This costs me £200.00, £350.00 or £450.00?

d. In a traffic census at an accident black spot, 938 cars are seen to pass in an hour. Of these, 300 are going at over 50 kilometres per hour (kph) and 638 are travelling more slowly. The average speed of these cars is calculated to be 36.4 kph, 51 kph or 16 kph?

3. Write the following to 3 decimal places:

1.9755 10,002.9752 209.452 12.73 0.00012456

4. Express the numbers in question **3** to 3 significant figures.

5. Express the following in scientific notation

12,000,000 0.00001254 9.999999999 999.999

6. Write the following numbers out in full

2.678×10^6 4.1×10^{-9} 1×10^0

Solutions:

1. 0.03 0.0025 $0.0\dot{6}$ (remember the $\dot{6}$ means that the digit 6 repeats forever – it's called 6 *recurring*).
2. Use your calculator to check **a.**, **b.** and **c.** In **d.** the only reasonable answer is 36.4.
3. 1.976 10,002.975 209.452 12.730 0.000
4. 1.98 10,000 209 12.7 0.000125
5. 1.2×10^7 1.254×10^{-5} 9.999999999×10^0 9.99999×10^2
6. 2,678,000 0.0000000041 1

Assessment 4

1. Evaluate the following using a calculator:

   $\dfrac{312 \times 1.01}{42.7 - 21.5}$ $432.2 - (543.2 - 10.17)/2.5$

2. Say which of the following results seem sensible and explain why. Do *not* use a calculator except to check your answer.
   - **a.** A leisure aircraft is owned by a Flying Group. The Group estimate that maintenance costs are likely to be £1000 next year. The main running cost is fuel which currently costs £0.7 per litre, but will be £0.95 per litre next year. Last year the aircraft did 9000 km. Average fuel consumption is 2.9 km per litre. If the 10 members of the Group are to share costs equally, the Treasurer suggests that a suitable monthly cost per member which ensures that maintenance and fuel costs are covered is £32.90, £44.12 or £12.10.
   - **b.** $\dfrac{111 + (20.5 \times 10.7)}{302} = 10.939,\ 1.0939\ \text{or}\ 0.10939$
   - **c.** I give myself a budget of \$100 a day for my 8-day holiday. At the start of my holiday I have £600 which I change into dollars at a rate of \$1.57 = £1. During the holiday I do not exceed my budget. At the end of the holiday I calculate that I should have \$40 or \$142 or \$10 left. (Ignore commission charges.)
3. Express the following to 4 decimal places

   13.66666 –3.156 200,000.00001 156.99999 55.12345

4. Express the numbers in question **3** to 2 significant figures.
5. Write the following in scientific notation

   3,000,051.0 0.0000009142 –102.01 4.14

6. Write the following numbers out in full

   3.42×10^8 1.004×10^{-6} 9.99×10^3

5 Introducing letters and symbols

test box 5

The total cost of a holiday for four friends comprises the cost of four return flights at $*f* each and a charge of $70 per day for a rented cottage. Write down an expression for the cost of the holiday for each person, if they go for *d* days.

Evaluate $\frac{2st}{s+1}$ when $s = 2$ and $t = 6$.

Evaluate 3a + *b*, when $a = 4$ and $b = 5$.

Simplify $\frac{a}{b} + 2 - 5a - 3\frac{a}{b} + a$

What is meant by 'modelling'? Have you ever used a spreadsheet?

Solutions:

$\frac{4f + 70d}{4}$ or $f + \frac{70d}{4}$, 8, 17, $-2\frac{a}{b} + 2 - 4a$

See the sub-sections on 'models' and 'using a spreadsheet' below.

Why use letters and symbols?

Suppose we have £200 and wish to purchase some euros. The exchange rate is £1 = 1.5 euros and for simplicity we will ignore any commission charge. Using a calculator we can work out that the £200 will buy 200 × 1.5 = 300 euros.

That's fine – but exchange rates fluctuate and not everyone wants to change exactly £200. It would be much more useful to develop a *general relationship* between the number of pounds, the exchange rate and the number of euros.

To obtain the figure of 300 euros we *multiplied* the number of pounds by the exchange rate so the relationship we seek is

$$\text{euros} = \text{pounds} \times \text{rate}$$

This relationship is valid for any number of pounds and any exchange rate. So to calculate, for instance, how many euros we would obtain when the exchange rate is 1.9 and we have £300 to spend we write out the relationship again:

$$\text{euros} = \text{pounds} \times \text{rate}$$

but *substitute* 300 instead of 'pounds' and 1.9 instead of 'rate' to give

$$\text{euros} = 300 \times 1.9$$

and so euros = 570. We would obtain 570 euros.

It is usual to use a single letter to represent each of the entities in a relationship. Letters like *x* and *y* are often used but sometimes we use 'meaningful' letters like *e* for euros, *p* for pounds and *r* for rate, in which case our relationship would be written

$$e = p \times r$$

However, it is common to omit the multiplication sign adjacent to symbols – just as we omit multiplication signs next to brackets – and so we could write this as

$$e = pr$$

A relationship like $e = pr$ is an *equation* (because it contains an = sign), or we could say that the *formula* for the number of euros, e, is pr.

Once we have an equation or formula it can be used in different ways to solve a variety of problems. For instance, suppose we know that we received 960 euros at the Bureau de Change in exchange for £800, but we can't remember what the exchange rate was. Substituting $e = 960$ and $p = 800$ into the equation above gives

$$960 = 800r$$

We now need to *solve* the equation for the unknown quantity r. We will see how to do this in Chapter EM3, but for now we will tell you that the solution is

$$r = \frac{960}{800} = 1.2$$

so the exchange rate was £1 = 1.2 euros.

This currency example is relatively uncomplicated but it has demonstrated that symbols are useful

(i) to represent the general relationship between the quantities of interest; and so
(ii) to enable calculation of one quantity from the others.

Using symbols to represent relationships

We now concentrate on how to turn information on the quantities of interest into expressions involving symbols. There is no magic way of doing this but if you find it hard we suggest that you break the task into two steps as follows.

1. Read through the information you have been given, but as you come across each quantity, assign it a symbol. Make a written note of these.
2. Read through again, but as you read 'translate' each fact you are given into symbols.

We include some examples with commentary. Remember that multiplication signs are usually omitted next to symbols.

check these ▷

1. It costs £2000 (the fixed cost) to set up a production run in a factory, and then a further £5 (the variable cost per unit) for each item manufactured. Write down an expression for the total production cost.

Solution:
On a first reading we realise that we need a symbol for the number of items manufactured (say, n) and a symbol for the total production cost (say, C).

During a second read-through we make the following jottings, which culminate in the desired expression for total production cost.

Fixed cost	2000
n items at a cost of £5 per item $5 \times n$	$5n$
Total cost $C =$	$2000 + 5n$

2. A restaurant has two menus, a tourist menu at £8 and a gourmet menu at £15. Write down a formula for the cost of the food for a party in which t customers have the tourist menu and g customers choose the gourmet menu.

Solution:
The symbols t and g have already been chosen for us here, but we adopt C for the total cost. Reading through we would write down something like

Tourist menu £8	t customers	$8t$
Gourmet £15	g customers	$15g$
Total cost is C =		$8t + 15g$

3. I want to organise a group of friends to hire a boat on the river for the day. The basic hire cost is £60, but we must also pay for fuel which costs £5 an hour. Write down an expression for the cost per person, C. (This might be of interest to establish how many friends I need to ask to keep the cost per person down to a particular amount.)

Solution:
Suppose I ask $n - 1$ friends so there are n of us altogether, and hire a boat for h hours. The total cost will be $60 + 5h$, so the cost per person will be

$$C = \frac{60 + 5h}{n}$$

Constants and variables

In many applications some amounts will be fixed – like the cost of fuel in **3** or the price of the meal in **2**. These fixed amounts are called *constants*. They are usually numbers, although a symbol which represents a particular value, like $\pi = 3.14159$ (from the formula for the circumference of a circle, $2\pi r$, where r is the radius) is also a constant.

On the other hand, symbols that represent quantities that can change, such as the number of people n, or the number of hours, h in **3**, are called *variables*.

Evaluating expressions

Once you have an expression you will often need to calculate its value when the variables in it take particular values.

For instance, suppose the amount of tax payable on a salary of S at a tax rate of t% is given by

$$\frac{(S - 3000)t}{100}$$

If Paul earns a salary of £10,000 and the tax rate is 30% then he must pay

$$\frac{(10{,}000 - 3000) \times 30}{100} = £2100$$

All we have done is replace the symbols in the formula with the values we are interested in. This is called *substitution*.

check this ▷

The amount of interest I receive on an investment of £P at r% interest over one year *less* a management charge of £10 + 0.01P is

$$\frac{P(r-1)}{100}-10$$

Calculate the amount of interest I would receive in the following cases

a. On an investment of £10,000 when the interest rate is 5%.

b. On an investment of £500 when the interest rate is 10%.

Solution:

a. $\frac{10,000\times(5-1)}{100}-10=£390$

b. $\frac{500\times(10-1)}{100}-10=£35$

Models

Much of the work you will do for your degree and beyond will require you to build financial, accounting and economic models. By a *model* we usually mean one or more equations which represent the real-life situation.

The advantage of a model is that it allows us to answer 'what if ...' type questions, without changing the real system. We illustrate this with a simple model for a clothing manufacturer's business.

Suppose it costs $10,000 to design a particular item of ladies clothing, and the unit production cost is $15. We might model the profit as

Profit = price × number sold – (10,000 + 15 × number manufactured)

The model enables the manufacturer to calculate the profit for a variety of values of the unknown variables (price, number sold and number manufactured) so that he may choose the values for price and number manufactured that give the highest profit. (He is not likely to have any control on the number sold.)

A model does not pretend to be exact. It will often be based on (and only as good as) a series of assumptions that the modeller makes (and should state), but the hope is that the model is a reasonable approximation to the real situation.

A model may be as complex or as simple as we like. For instance, the model above makes the assumption that the number sold is unaffected by the price. It would probably be more realistic to assume that as the price increased the number of sales decreased. This could be built into the model by introducing another relationship, for instance

number sold = 1000 – (10 × price)

and the two relationships could be used together to find the best price and number to manufacture.

Using a spreadsheet

For those of you who are *not* familiar with spreadsheets, they are a type of computer software that comprises a grid of numerical quantities and the relationship between them. They have become a useful tool for building models in accountancy, business, finance and elsewhere.

The rows of a spreadsheet are numbered **1**, **2**, **3**, ..., etc. and columns are labelled **A**, **B**, **C**, ..., etc. so that each cell of the grid is uniquely identified by a letter and a number like **A1**, **B23** or **E2**.

Into a cell of a spreadsheet the user can either enter a number – in which case the cell always takes that value – or an expression giving the relationship between the current cell and the others. For instance, if I would like the total of cells **E12** and **E13** to appear in cell **E14** I would enter the expression

= E12 + E13

into cell **E14**. The exact format of this expression may differ depending on the spreadsheet package, but the principle is always the same.

Spreadsheets are programmed so that when a value in a cell is changed, all the values elsewhere in the spreadsheet, which are influenced by that cell's value, will be changed automatically. Thus, 'what if ...' questions can be answered by trial and error.

The spreadsheet screen can show two modes – one which shows the symbolic expressions in each cell and the other which gives the numerical values implied by these.

A simple spreadsheet which calculates student marks is shown below.

	A	B	C	D	E	F
1	student	c w 1 %	c w 2 %	overall c w	exam %	overall %
2	Catherine	58	66	= (B2+C2)/2	77	= 0.2*D2+0.8*E2
3	Sylvia	82	70	= (B3+C3)/2	45	= 0.2*D3+0.8*E3
4	Malcolm	55	45	= (B4+C4)/2	76	= 0.2*D4+0.8*E4
5	Dennis	78	70	= (B5+C5)/2	60	= 0.2*D5+0.8*E5
6	Veronica	90	82	= (B6+C6)/2	74	= 0.2*D6+0.8*E6
7	Gillian	60	62	= (B7+C7)/2	58	= 0.2*D7+0.8*E7
8						
9					average	= (F2+F3+F4+F5+F6+F7)/6

20% of the marks on a Maths course can be gained from coursework and 80% from the exam. The two pieces of coursework carry equal weight and are each marked out of 100. A spreadsheet which calculates and displays the overall coursework mark and the overall marks of each student, and calculates the class's average mark is shown above in formula mode. Notice that the expressions for the overall coursework marks require brackets.

The same spreadsheet in 'values' mode is shown below.

student	c w 1 %	c w 2 %	overall c w	exam %	overall %
Catherine	58	66	62	77	74
Sylvia	82	70	76	45	51.2
Malcolm	55	45	50	76	70.8
Dennis	78	70	74	60	62.8
Veronica	90	82	86	74	76.4
Gillian	60	62	61	58	58.6
				average	65.63

In this book we will make use of a spreadsheet program, Microsoft Excel, particularly for *Statistics* and *Business Modelling*.

Work card 5

1. Write down expressions for the following. Start by naming the variables you need.
   - **a.** The cost of an m metre length of a roll of carpet when the width of the roll of carpet is 4 m and the cost per square metre is £12.
   - **b.** The cost of a group's meal at a pizzeria, when pizzas are £4 each and bottles of wine £7. No other food or drink is available.
   - **c.** The amount a gardener earns in a week when he charges £5 per hour for manual work like weeding but £8 per hour for skilled gardening work.
   - **d.** The net amount given or charged to me in one month by my bank when they give me 5% interest on my average balance over the month, but charge me 30p per debit transaction.
   - **e.** The cost of petrol for a journey of L miles when my average fuel consumption is 30 miles per gallon and the cost of fuel is £p per litre. Assume there are 4.54 litres in a gallon.

2. Salaries in a computer consultancy firm are calculated according to the following formula.

$$S = 8000 + 300(A - 20) + 1000Y$$

   where A is the age of the employee in years, and Y is the number of years of experience they have in computing.

   What will an employee's salary be if she
   - **a.** Has just graduated at 21 with no work experience?
   - **b.** Left school at 16 and has worked in computing for 4 years until today?
   - **c.** Is now 60, but took up Computing work at the age of 52 for the first time?

3. A company has lent a sum of £P. It is to be repaid at r% interest in one year's time. The company calculates that the prevailing market interest rate is i%, so the present value to the company of the repayment is

$$V = \frac{P(1+\frac{r}{100})}{1+\frac{i}{100}}$$

   Find the present value in each of the following cases:
   - **a.** The loan is £10,000 at 8% interest and the market interest rate is 10%.
   - **b.** The loan is £15,000 at 12% interest and the market interest rate is 10%.
   - **c.** The loan is £20,000 at 5% interest and the market interest rate is 5%.

Solutions:

1. a. $48m$ **b.** $4p + 7w$ **c.** $5m + 8s$ **d.** $0.05A - 0.3D$

e. $\frac{L}{30} \times 4.54p$ (L is length of journey in miles)

2. a. 8300 **b.** 12,000 **c.** 28,000

3. a. £9818.18 **b.** £15,272.72 **c.** £20,000.

Assessment 5

1. Write down an expression for the floor area of a two-room flat when the first room has one and a half times the width and twice the breadth as the second. Start by naming the variables you are going to use.

2. I want to buy some Botswana pulas and the exchange rate is 8.25 BWP = £1 if I buy them abroad using a credit card or 8.5 BWP = £1 for a cash transaction at a Bureau de Change. The credit card company will charge me 2.3% commission on a credit card transaction whereas a Bureau de Change charges 2% commission plus a fee of £3.

   Write down an expression for the cost (including commission) of buying a particular number of Botswana pulas **(a)** by credit card and **(b)** from a Bureau de Change.

   (i) Evaluate each of these expressions for 700 BWP, 800 BWP and 1000 BWP respectively.

   (ii) Hazard a guess as to the number of pulas you would have to buy for the cost to be the same whether you use a credit card or cash. (To be continued later!)

3. A leisure aircraft is owned by the Broadland Flying Group which has 10 members. The costs of keeping the aircraft comprise maintenance costs (including insurance) and the running cost which is mainly fuel. Average fuel consumption is 9 miles per gallon.

   If the 10 members of the Group are to share costs equally, write down a formula to enable Gordon the treasurer to calculate the annual cost for each member if the aircraft travels n miles in a year, incurs maintenance costs of £M and fuel costs £3 a gallon.

4. Consider the model for the clothing manufacturer

   Profit = price × number sold – (10,000 + 15 × number manufactured).

   Set up a spreadsheet to investigate the profit, when the selling price is £18, for a range of values for the number sold and the number manufactured. Comment on your results. Adapt your spreadsheet to investigate how many items must be manufactured in order to break even (make a profit of 0) assuming that the number sold is the same as the number manufactured.

6 Working with symbols: adding, subtracting, multiplying, dividing

test box 6

Simplify the following by collecting like terms:

$3pq + 2q + 2pq$

Write the following expressions more succinctly:

$3 \times (-2p) \times 2 \times p$ $\quad$ $2 \cdot 3 \cdot 5$

Solutions:

$5pq + 2q$

$-12p^2$ $\quad$ 30

To construct models and solve equations we need to know how to manipulate symbols. This need not be a problem if we remember that the symbols merely *stand in for the numbers* and so can be treated in exactly the same way. In this section we recall the work of Sections 1–3, but apply it particularly to expressions containing symbols.

Negative symbols

The rules for adding, subtracting, multiplying and dividing numbers apply equally well to symbols. We can summarise these rules as

Adding and subtracting

SAME SIGNS

– (–a) or **+** (+a) gives **+a**

OPPOSITE SIGNS

+ (–a) or **–** (+a) gives **–a**

where **a** is any number, symbol or expression.

Multiplying and dividing

SAME SIGNS

multiplying $a \times b = ab$

$(-a)(-b) = ab$

dividing $a \div b = \frac{a}{b}$

and $(-a) \div (-b) = \frac{-a}{-b} = \frac{a}{b}$

OPPOSITE SIGNS

multiplying $a \times (-b) = -ab$

$(-a) \times b = -ab$

dividing $(-a) \div b = \frac{-a}{b} = -\frac{a}{b}$

$a \div (-b) = \frac{a}{-b} = -\frac{a}{b}$

check these ▷

$2 + (-a) = 2 - a$

$3 - (-a) = 3 + a$

$2 \times (-a) = -2a$

$(-3) \times (-a) = 3a$

$(-b) \times c = -bc$

$(-c) \times (-a) = ac$

$\frac{p}{-q} = -\frac{p}{q}$ $\frac{a}{a} = 1$ $\frac{-x}{-x} = 1$ $\frac{-a}{a} = -1$

Addition and subtraction: collecting 'like terms'

Because $a + a + 2a$ means one 'a' plus one 'a' plus two 'a's the terms can be collected together and written as $4a$. We can do this because each term is the same apart from the number – called the *coefficient*, which it is multiplied by. It is just like saying that 'one banana plus another banana plus another two bananas gives four bananas'.

Even when we have more complicated terms we can collect them together *as long as they are the same*. For instance

$$2pq + pq - 5pq$$

simplifies to $-2pq$, or

$$\frac{s}{2r} + 4\frac{s}{2r}$$

is equivalent to

$$5\frac{s}{2r}$$

as each term is so many

$$\frac{s}{2r}\text{s}$$

Notice, however, that we *can't* collect together any terms in

$$3pq + p + q$$

as they are all multiples of different things – we can't add 3 bananas, an orange and a grapefruit!

Often just some of the terms in an expression can be collected together. For instance,

$$pq + 2p + 5pq$$

simplifies to $6pq + 2p$.

check these ▷

Where possible, simplify the following:

$5rs - 3rs + rs$

All the terms involve a number of rss, so the expression simplifies to $3rs$.

$9pq + 4q - 5pq - q$

The pq terms can be collected to give $9pq - 5pq = 4pq$, but the terms in q must be collected separately to give $4q - q = 3q$. So the expression simplifies to $4pq + 3q$.

$$3\frac{a}{2} - 2\frac{a}{2} + 5\frac{a}{2}$$

All the terms involve a number of $\frac{a}{2}$s, so the expression simplifies to $6\frac{a}{2}$. We will see later that this can be simplified further to $3a$.

$$2xz + 5zx - 3z$$

You should spot here that xz is the same as zx (as the order in which we multiply doesn't effect the result), so we can collect together $2xz + 5zx$ to give $7xz$. The whole expression simplifies to $7xz - 3z$.

In the last example it would have been easier to spot that $2xz$ and $5zx$ could be collected together if the xz and zx terms had been written in the same way. It is therefore a good idea to write letters that are multiplied together in alphabetical order. This is one of the suggestions below.

Some conventions for multiplying

1. We have already said that there is no need to include a multiplication sign '×' next to a bracket or next to a symbol. As a consequence, whenever letters or brackets appear adjacent to each other a multiplication sign is implied. For instance $2ab(c + d)$ means $2 \times a \times b \times (c + d)$.
2. We can't just drop the multiplication sign when multiplying two or more numbers as we wouldn't be able to tell when one number finished and the next one started. For example 357890 might mean 35×7890 or $3 \times 57 \times 890$ or many other possibilities. Instead we can shorten expressions by writing a slightly raised *dot* instead of the × sign. So $357 \times 8 \times 90$ could be written $357 \cdot 8 \cdot 90$.
3. When a number is multiplied by a letter it is conventional to write the number first so, for instance, we write $5n$ instead of $n5$.
4. When a mixture of numbers and symbols are multiplied together, it is more succinct to multiply the numbers together so, for instance, instead of $4 \times p \times q \times 2$ we would write $8pq$.
5. When several symbols are multiplied together it is usual to write them in alphabetical order. This means that 'like' terms can be spotted easily. So $cad \times 2$ would more usually be written $2acd$.
6. When a number or symbol is multiplied by itself we say it is 'squared', and we usually write the number with a 2 superscript. For instance, 3^2 is called 'three squared' and means 3×3. In a similar way $(ab)^2$ means $(ab) \times (ab)$.

check this ▷

Check that you understand what the following expressions mean

1. $3p(2 + a)$ $\quad \dfrac{100}{rc}$ $\quad \dfrac{2 \cdot 3 \cdot 4}{ax}$ $\quad \dfrac{100pq}{3 \cdot 2 \cdot 1}$ $\quad \dfrac{a^2b}{c^2}$

Write the following expressions more succinctly

2. $y \times z \times 2 \times a$ $\quad 3 \times b \times a \times 2$ $\quad a \times 2b \times 5$

Solutions:

1. Writing out these expressions in long-hand using × for multiplication gives: $3 \times p \times (2 + a)$, $100 \div (r \times c)$, $(2 \times 3 \times 4) \div (a \times x)$, $(100 \times p \times q) \div (3 \times 2 \times 1)$, $(a \times a \times b) \div (c \times c)$
2. $2ayz$, $6ab$; remember that this is $a \times 2 \times b \times 5$, giving $10ab$.

Can I do this?

There are many more techniques for working with symbols and some will be considered in Chapter EM2. However, if you are in doubt as to whether two expressions are equivalent you can always evaluate both of them for some arbitrarily chosen numbers and see if the results are the same.

For instance, suppose you are unsure whether $p \times q$ is the same as $q \times p$. (We have picked an easy one to start with – we have already said that this is true.) If, for example, $p = 2$ and $q = 1$, then $p \times q = 2 \times 1 = 2$ and $q \times p = 1 \times 2 = 2$, so the expressions are equal for these values. Now try $p = -5$ and $q = 2$, putting a negative number to the test, and we have $p \times q = -5 \times 2 = -10$ and $q \times p = 2 \times -5 = -10$ which again are equal.

Be warned, however, that 'trying out' values like this does not constitute proof that the expressions are equal – you may just have been lucky and by chance selected values that worked. If the two expressions are *not* equal for the values you have chosen, then this does, however, prove that they are *not* equivalent expressions.

For instance, is it true that

$$\frac{m+n}{n}$$

is the same as m? Try some values for m and n to see whether this is true or not before you read on.

check this ▷

Is $\frac{m+n}{n}$ the same as m?

Solution:

We'll choose $m = 100$, $n = 5$ first. When we evaluate the two expressions we obtain

$$\frac{m+n}{n} = \frac{105}{5} = 21$$

whereas $m = 100$. No, this does *not* work. We don't need to try any more values for m and n as we only need to find one counter-example to show that the two expressions are not equal.

Work card 6

1. Simplify, by collecting like terms where possible

a. $b + b + b + a + a$

b. $p + q + 2pq - q + 3qp$

c. $2p + 2q + p + pqr - 2pqr$

d. $2\frac{x}{y} + x - 2x + 3\frac{x}{y}$

2. Write the following expressions more succinctly

a. $2 \times n \times 3 \times m$ **b.** $pq \times rs \times 3$

c. $a \cdot 2 \cdot 10 \cdot zb$ **d.** $(z \times y) \times (2 \times w)$

3. You dimly recollect from your school career that it is valid to 'cancel down' and that

$$\frac{2a}{2b} \text{ is the same as } \frac{a}{b}.$$

Try out some numbers to see if this seems to be the case or not.

4. Is it true that

$$\frac{1+2a}{2b} = \frac{1+a}{b}?$$

Solutions:

1. a. $3b + 2a$ **b.** $p + 5pq$ **c.** $3p + 2q - pqr$ **d.** $5\frac{x}{y} - x$

2. a. $6mn$ **b.** $3pqrs$ **c.** $20abz$ **d.** $2wyz$

3. Try for instance $a = 5$ and $b = 10$, $2a = 10$ and $2b = 20$, so

$$\frac{2a}{2b} = \frac{10}{20} = 0.5 = \frac{a}{b}$$

so it works. Now try one or more negative numbers, say $a = -1$ and $b = 3$; then

$$\frac{2a}{2b} = \frac{-2}{6} = -\frac{1}{3} \text{ and } \frac{a}{b} = \frac{-1}{3} = -\frac{1}{3}$$

so again it works. In fact it is true and we cover it in the next section!

4. No it is not true. Try for instance $a = 1$, $b = 2$; then

$$\frac{1+2a}{2b} = \frac{3}{4} \text{ whereas } \frac{1+a}{b} = \frac{2}{2} = 1$$

which is clearly different.

Assessment 6

1. Simplify by collecting like terms:

a. $3x - 2xy + 3y + 3xy$

b. $p - 2q - 2q^2 + q^2 - 3p^2$

c. $3 \cdot \frac{a}{2b} + a - 2\frac{a}{2b} + 3a$

2. Write down the following expressions more succinctly:

a. $\frac{f \times b \times 2 \cdot 3}{2 \times e \times d}$ **b.** $(c \times b) \times (a \times d) \times 3$

3. Investigate whether $a(b + d)$ is the same as $ab + d$.