

1. DATA ANALYSIS

Uncertainties in calculations

Rule 1 When two measurements are added or subtracted, the uncertainty in the result is always equal to the sum of the uncertainties of the two measurements.

Measurement 1 ; $d_1 \pm \Delta d_1$, where Δd_1 is the uncertainty of the measurement,

Measurement 2 ; $d_2 \pm \Delta d_2$, where Δd_2 is the uncertainty of the measurement,

a) The **difference** between the two measurements = $(d_1 \pm \Delta d_1) - (d_2 \pm \Delta d_2)$
= $(d_1 - d_2) \pm (\Delta d_1 + \Delta d_2)$

This is because the difference can range from

- a maximum value of $(d_1 + \Delta d_1) - (d_2 - \Delta d_2) = (d_1 - d_2) + (\Delta d_1 + \Delta d_2)$, to
- a minimum value of $(d_1 - \Delta d_1) - (d_2 + \Delta d_2) = (d_1 - d_2) - (\Delta d_1 + \Delta d_2)$,

The uncertainty is therefore $\Delta d_1 + \Delta d_2$

b) The **sum** of the two measurements = $(d_1 \pm \Delta d_1) + (d_2 \pm \Delta d_2)$
= $(d_1 + d_2) \pm (\Delta d_1 + \Delta d_2)$

This is because the sum can range from

- a maximum value of $(d_1 + \Delta d_1) + (d_2 + \Delta d_2) = (d_1 + d_2) + (\Delta d_1 + \Delta d_2)$, to
a minimum value of $(d_1 - \Delta d_1) + (d_2 - \Delta d_2) = (d_1 + d_2) - (\Delta d_1 + \Delta d_2)$,

The uncertainty is therefore $\Delta d_1 + \Delta d_2$

Rule 2 When two measurements are multiplied or divided , the percentage uncertainty in the result is always equal to the sum of the percentage uncertainties of the two measurements.

Measurement 1 ; $d_1 \pm \Delta d_1 = d_1 (1 \pm \frac{\Delta d_1}{d_1})$, where $\frac{\Delta d_1}{d_1}$ (x 100%) is the percentage uncertainty of the measurement ,

Measurement 2 ; $d_2 \pm \Delta d_2 = d_2 (1 \pm \frac{\Delta d_2}{d_2})$, where $\frac{\Delta d_2}{d_2}$ (x 100) is the percentage uncertainty of the measurement ,

$$\begin{aligned}
\text{a) The product of the two measurements} &= d_1 \left(1 \pm \frac{\Delta d_1}{d_1} \right) \times d_2 \left(1 \pm \frac{\Delta d_2}{d_2} \right) \\
&= d_1 d_2 \left(1 \pm \frac{\Delta d_1}{d_1} \right) \times \left(1 \pm \frac{\Delta d_2}{d_2} \right) \\
&= d_1 d_2 \left(1 \pm \frac{\Delta d_1}{d_1} \pm \frac{\Delta d_2}{d_2} + \frac{\Delta d_1}{d_1} \frac{\Delta d_2}{d_2} \right) \\
&= d_1 d_2 \left(1 \pm \frac{\Delta d_1}{d_1} \pm \frac{\Delta d_2}{d_2} \right)
\end{aligned}$$

as the last term in the bracket above is negligible compared with the other terms .

$$\therefore \text{ The percentage uncertainty in the product } d_1 d_2 \text{ is } \frac{\Delta d_1}{d_1} + \frac{\Delta d_2}{d_2} \text{ (x 100 \%)}$$

$$\begin{aligned}
\text{a) The quotient of the two measurements} &= \frac{d_1 \left(1 \pm \frac{\Delta d_1}{d_1} \right)}{d_2 \left(1 \pm \frac{\Delta d_2}{d_2} \right)} \\
&= \frac{d_1}{d_2} \left(1 \pm \frac{\Delta d_1}{d_1} \right) \left(1 \pm \frac{\Delta d_2}{d_2} \right)
\end{aligned}$$

To progress further with this analysis , we need to use the mathematical approximation

$$\frac{1}{(1 + y)} = (1 - y) \text{ for } y \ll 1$$

This is valid provided $y \ll 1$ because $(1 + y)(1 - y) = 1 + y - y + y^2 \approx 1$ (as y^2 is negligible)

$$\begin{aligned}
\therefore \frac{d_1 \left(1 \pm \frac{\Delta d_1}{d_1} \right)}{d_2 \left(1 \pm \frac{\Delta d_2}{d_2} \right)} &= \frac{d_1 \left(1 \pm \frac{\Delta d_1}{d_1} \right)}{d_2} \times \left(1 \pm \frac{\Delta d_2}{d_2} \right) \\
&= \frac{d_1}{d_2} \left(1 \pm \frac{\Delta d_1}{d_1} \pm \frac{\Delta d_2}{d_2} - \frac{\Delta d_1}{d_1} \frac{\Delta d_2}{d_2} \right) \\
&= \frac{d_1}{d_2} \left(1 \pm \frac{\Delta d_1}{d_1} \pm \frac{\Delta d_2}{d_2} \right) \text{ as the last}
\end{aligned}$$

term in the bracket above is negligible compared with the other terms .

$$\therefore \text{ The percentage uncertainty in the quotient } \frac{d_1}{d_2} \text{ is } \frac{\Delta d_1}{d_1} + \frac{\Delta d_2}{d_2} \text{ (x 100 \%)}$$

Further note

Where a measured quantity y is raised to a power n ,

the percentage uncertainty in $y^n = n$ times the percentage uncertainty of y .

This follows because $y^n =$ the product of y with itself n times so the percentage uncertainty of y must be added to itself n times to give the percentage uncertainty in y^n .

Examples

1. The diameter of a wire was measured at 0.28 ± 0.01 mm. Calculate

- the percentage uncertainty in the diameter of the wire,
- the area of cross-section of the wire,
- the percentage uncertainty in the area of cross section of the wire.

Solution

(a) % uncertainty in diameter $d = \frac{0.01}{0.28} \times 100 = 3.6\%$

(b) The area of cross-section $A = \frac{\pi d^2}{4} = \frac{\pi (0.28)^2}{4} \text{ mm}^2 = 6.16 \times 10^{-2} \text{ mm}^2$

(c) % uncertainty in $A = 2 \times$ % uncertainty in d because $A = \frac{\pi d^2}{4}$

\therefore % uncertainty in $A = 7.2\%$

2. The mass and length of a certain length of the above wire was measured as below :-

$$\text{Mass} = 0.496 \pm 0.002 \text{ grams}, \text{ Length} = 1000 \pm 2 \text{ mm}$$

Calculate

- the density of the wire in kg m^{-3}
- the percentage uncertainty of the density
- the uncertainty of the density

Hence state the calculated density value including the uncertainty of this value.

Solution

(a) Area of cross-section, $A = 6.16 \times 10^{-2} \times 10^{-6} \text{ m}^2$

\therefore Volume, $V = \text{length} \times \text{area of cross-section} = 1.00 \times 6.16 \times 10^{-8} \text{ m}^3$

$$\text{Mass, } m = 0.496 \times 10^{-3} \text{ kg}$$

$$\therefore \text{Density} = \frac{\text{Mass}}{\text{Volume}} = \frac{0.496 \times 10^{-3} \text{ kg}}{6.16 \times 10^{-8} \text{ m}^3} = 8050 \text{ kg m}^{-3}$$

$$(b) \text{ \% uncertainty in length } L = \frac{2}{1000} \times 100 = 0.2 \%$$

Because the volume $V = \text{length } L \times \text{area of cross-section } A$,

$$\text{\% uncertainty in volume } V = \text{\% uncertainty in length} + \text{\% uncertainty in } A = 7.4\%$$

$$\text{\% uncertainty in mass } m = \frac{0.002}{0.496} \times 100 = 0.4 \%$$

$$\therefore \text{\% uncertainty in density} = \text{\% uncertainty in mass} + \text{\% uncertainty in volume} = 7.8\%$$

$$(c) \text{ Uncertainty of the density} = 7.8 \% \text{ of } 8050 \text{ kg m}^{-3} = 628 \text{ kg m}^{-3}.$$

$$\therefore \text{Density} = 8050 \pm 630 \text{ kg m}^{-3}$$