Contents

CHAPTER 1

Two-Currency, Three-Currency and Multi-Currency Arbitrage

1.1 DEFINITION OF ARBITRAGE

Arbitrage is generally defined as capitalising on a discrepancy in quoted prices, triggered by the violation of an equilibrium (pricing) condition. It is often the case that arbitrage is portrayed to be a riskless operation, in the sense that all of the decision variables are known when the decision is made, but the process invariably involves risk, such as the risk of non-delivery (Herstatt risk). The arbitrage process restores equilibrium via changes in the supply of and demand for the underlying commodity, asset or currency. These changes in supply and demand cause price changes in such a way as to restore the equilibrium no-arbitrage condition. At this point the arbitrage process comes to an end, as the operation becomes unprofitable.

In the special case of the foreign exchange market, arbitrage is defined as the simultaneous purchase and sale of currencies for the sake of making profit. Profitable arbitrage opportunities arise in the spot or forward foreign exchange market either because exchange rates differ from one financial centre to another or because they are inconsistent, violating an equilibrium pricing condition in both cases. It must be mentioned at the outset that in today's integrated financial markets, arbitrage opportunities of this kind rarely, if at all, arise. And even if they arose, they would be quickly exploited by arbitragers to the point of "extinction". It is, however, still important to study these operations because they provide the mechanisms whereby the equilibrium conditions are maintained. In fact, the no-arbitrage condition is typically taken to define the equilibrium price of the underlying asset(s), and hence the study of arbitrage boils down to the study of price determination in financial markets, which is a crucial element of financial economics. At a later stage, we shall challenge some of the misconceptions about arbitrage that arise from its conventional definition as stated earlier.

1.2 TWO-CURRENCY ARBITRAGE

Also known as spatial, locational or two-point arbitrage, two-currency arbitrage opportunities arise when the exchange rate between two currencies is not the same in two financial centres. Although two-currency arbitrage can be conducted in both the spot and forward markets, the discussion will be limited to the spot market.

Let us assume that there are two financial centres, A and B, and two currencies, x and y, and that (for the time being) there are no transaction costs (for example, brokerage fees), no taxes and a zero bid–offer spread. If $S(x/y)$ is the spot exchange rate between x and y measured as the price (in terms of x) of one unit of y, then two-currency arbitrage will be triggered if

$$S_A(x/y) \neq S_B(x/y) \tag{1.1}$$

which means that the exchange rate between the two currencies has two different values in two financial centres at the same point in time. To simplify the notation, we will for the rest of this section drop the units of measurement, (x/y), from the exchange rate symbol. It is essential, however, to bear in mind that the exchange rate is measured as the price of one unit of y, $S(x/y)$, and not the other way round.

If the arbitrage condition represented by (1.1) is violated, then one possibility is that

$$S_A > S_B \tag{1.2}$$

which means that currency y is cheaper in B than in A (or that currency x is cheaper in A). Two-currency arbitrage, in this case, would take the form of buying y where it is cheap (in B) and selling it where it is expensive (in A). The profit realised from this operation, π, is the difference between the selling and buying rates, or

$$\pi = S_A - S_B \tag{1.3}$$

Figure 1.1 shows how arbitrage affects the forces of supply and demand, and hence the exchange rates in both centres. As the demand for y rises in B, S_B rises, and as the supply of y rises in A, S_A falls, reducing arbitrage profit. This operation continues until profit declines to zero ($\pi = 0$). Thus the no-arbitrage condition, which is obtained when arbitrage profit falls to zero, is given by

$$S_A = S_B \tag{1.4}$$

Hence, any violation of the condition represented by (1.4) triggers (profitable) arbitrage. The no-arbitrage condition is represented by the points falling on the no-arbitrage line, which is a 45° line passing through the origin (Figure 1.2). Points falling off the line represent violation of the no-arbitrage condition. Those falling above the line indicate a violation of the no-arbitrage condition as in (1.2). Arbitrage causes a movement towards the line either by an

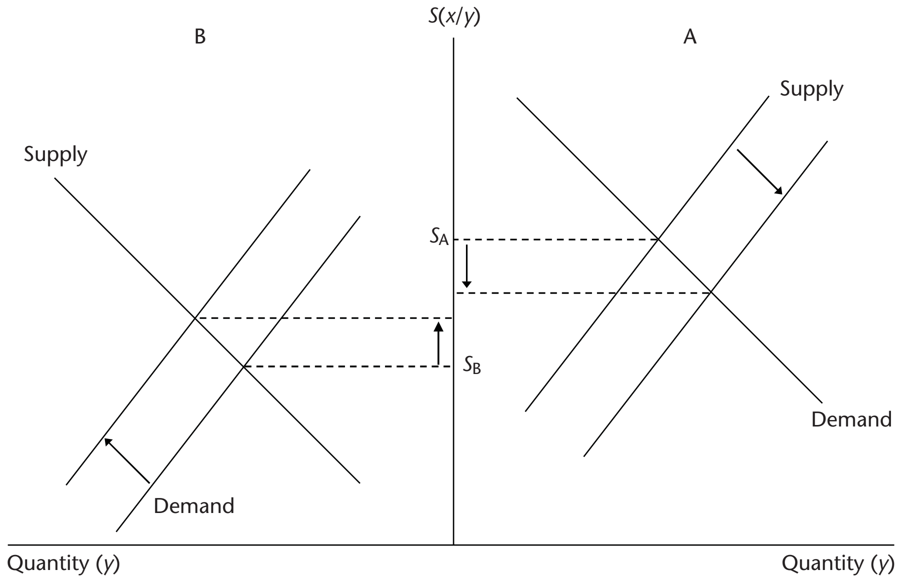

FIGURE 1.1 The effect of two-currency arbitrage.

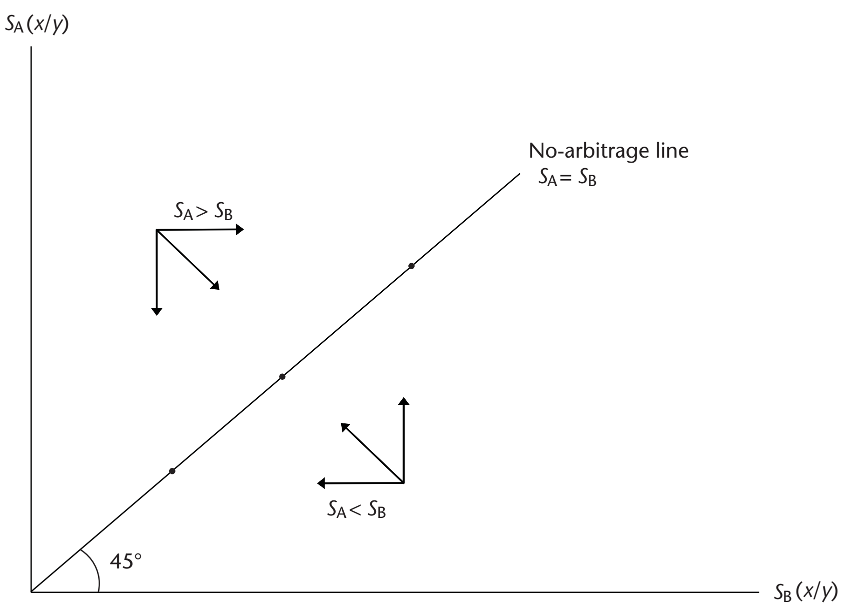

FIGURE 1.2 The no-arbitrage line (two-currency arbitrage).

increase in S_B, a decrease in S_A, or (more likely) both. Points below the line indicate a violation of the form $S_A < S_B$. In the following subsections some real-life complications are introduced.

Two-currency arbitrage with brokerage fees

Let us now assume that buyers and sellers have to pay brokerage fees on the transactions involving the buying and selling of currencies. Assume initially that brokerage fees are fixed and independent of the size of the transactions. Suppose now that the arbitrager wants to make profit by buying y in B and selling it in A. In the presence of fixed brokerage fees, the profit realised from this operation is

$$\pi = S_A - S_B - (\beta_A + \beta_B) \tag{1.5}$$

where β_A and β_B are the brokerage fees in financial centres A and B respectively. For the arbitrage operation to be profitable in this case the following condition must be satisfied

$$S_A - S_B > (\beta_A + \beta_B) \tag{1.6}$$

which means that the difference between the selling and buying rates must be greater than the sum of the brokerage fees incurred in the buying and selling transactions. Hence the no-arbitrage condition in the presence of fixed brokerage fees is given by

$$S_A = S_B + (\beta_A + \beta_B) \tag{1.7}$$

Likewise, if the arbitrager is to make profit by buying y in A and selling it in B then the following condition must be satisfied

$$S_B - S_A > (\beta_A + \beta_B) \tag{1.8}$$

and the no-arbitrage condition becomes

$$S_A = S_B - (\beta_A + \beta_B) \tag{1.9}$$

Figure 1.3 shows the effect of two-currency arbitrage in the presence of fixed brokerage fees when the arbitrager buys y in B and sells it in A. Demand increases in B and supply increases in A, leading to a rise in the exchange rate in B and a fall in A. In this case, however, arbitrage does not come to an end when the exchange rates are equal in the two financial centres, but when the difference between them is equal to the sum of brokerage fees incurred in both financial centres, $(\beta_A + \beta_B)$. Figure 1.4 shows what happens to the no-arbitrage line when there are fixed brokerage fees. A band, $2(\beta_A + \beta_B)$ wide, will be created around the original no-arbitrage line. The upper limit of the band is defined by equation (1.7), whereas the lower band is defined by equation (1.9). Points within the band but off the original no-arbitrage line indicate that while the exchange rates are not equal across financial centres, arbitrage is not profitable because arbitrage profit will be consumed by brokerage fees. Points falling outside the band define profitable arbitrage operations. Above the upper limit, arbitrage is profitable by buying y in B and selling it in A. Below the lower limit, arbitrage is profitable by buying y in A and selling it in B.

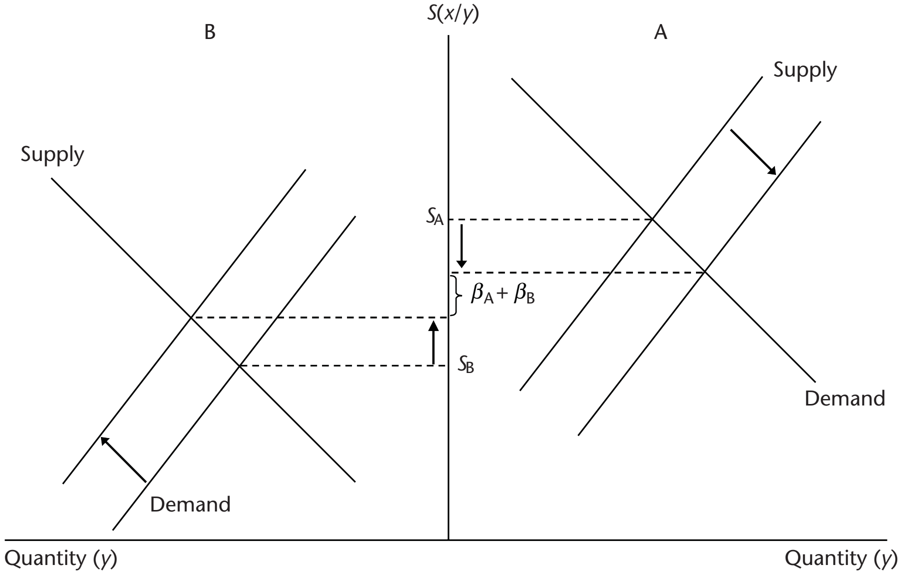

FIGURE 1.3 The effect of two-currency arbitrage in the presence of brokerage fees.

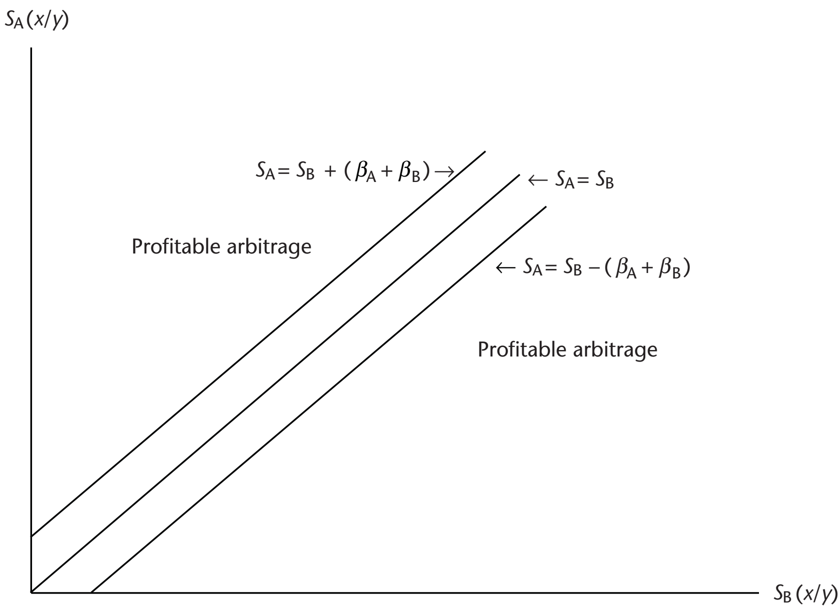

FIGURE 1.4 The no-arbitrage line in the presence of brokerage fees.

Assume now that brokerage fees depend on the size of the transactions, such that they are charged at the rates of β_A and β_B in financial centres A and B respectively. If $S_A > S_B$, then arbitragers will buy y in B and sell it in A. In this case the profit realised from arbitrage is

$$\pi = S_A(1-\beta_A) - S_B(1+\beta_B) \tag{1.10}$$

which means that arbitrage will be profitable ($\pi > 0$) if

$$S_A > S_B\left[\frac{1+\beta_B}{1-\beta_A}\right] \tag{1.11}$$

Alternatively, if $S_A < S_B$, then arbitragers will buy y in A and sell it in B. In this case the profit realised from arbitrage is

$$\pi = S_B(1-\beta_B) - S_A(1+\beta_A) \tag{1.12}$$

which means that arbitrage will be profitable if

$$S_A < S_B\left[\frac{1-\beta_B}{1+\beta_A}\right] \tag{1.13}$$

Hence the no-arbitrage lines associated with (1.11) and (1.13) respectively are

$$S_A = S_B\left[\frac{1+\beta_B}{1-\beta_A}\right] \tag{1.14}$$

and

$$S_A = S_B\left[\frac{1-\beta_B}{1+\beta_A}\right] \tag{1.15}$$

Figure 1.5 shows what happens to the no-arbitrage line in this case. Notice that since $0 < \beta_A < 1$ and $0 < \beta_B < 1$, it follows that $(1+\beta_B)/(1-\beta_A) > 1$, while $(1-\beta_B)/(1+\beta_A) < 1$. Diagrammatically, equations (1.14) and (1.15) are represented in Figure 1.5 by two lines intersecting with the original no-arbitrage line at the origin, with one being steeper (1.14) and the other flatter (1.15). Points within the triangular area define unprofitable arbitrage, as any profit realised from the difference in the exchange rates across the financial centres will be consumed by brokerage fees. Any point above or below the triangular area indicates a profitable arbitrage opportunity.

Two-currency arbitrage in the presence of taxes

Here we consider two kinds of tax: capital gains tax and Tobin tax. We start with the former. Suppose that capital gains tax is imposed on the profits realised from two-currency arbitrage in the financial centre where the profit is realised. It is easy to show that the presence of capital gains tax has no effect on the no-arbitrage line, because the only effect of the tax is to reduce the profit received by the arbitrager. As long as there is a discrepancy between the exchange rates, arbitrage will be profitable, though less so than in the absence of the tax. Arbitrage will not come to an end unless the discrepancy between the exchange rates disappears.

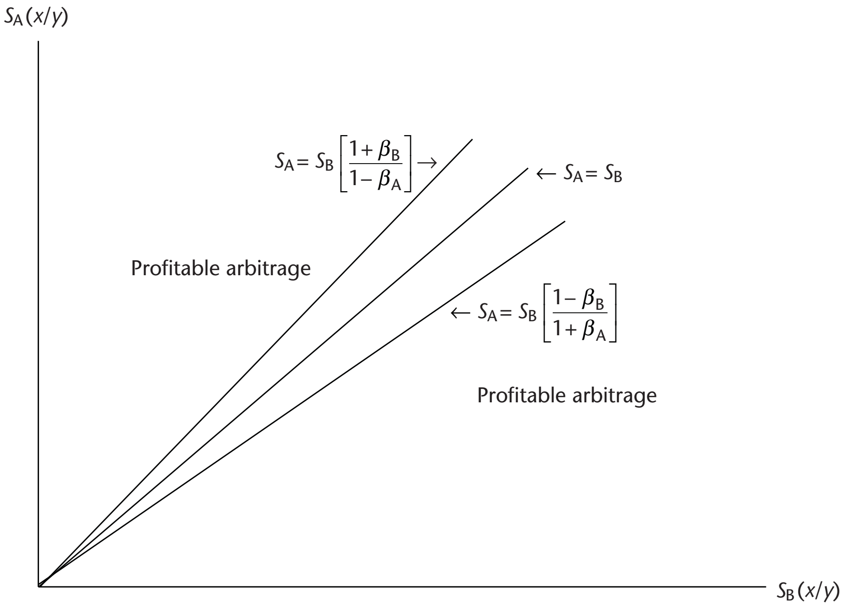

FIGURE 1.5 The no-arbitrage zone in the presence of progressive brokerage fees.

If the rate of capital gains tax is τ, then the after-tax arbitrage profit obtained by buying y in B and selling it in A is

$$\pi = (1-\tau)[S_A - S_B] \tag{1.16}$$

in which case the no-arbitrage line is given by

$$(1-\tau)[S_A - S_B] = 0 \tag{1.17}$$

which is equivalent to (1.4). Notice that $\partial\pi/\partial\tau < 0$.

Tobin tax was suggested by a Noble laureate, James Tobin, as a measure that would reduce the volatility in the foreign exchange market. It is imposed as a percentage of the value of the transaction, and hence it has the same effect as imposing brokerage fees on the buying and selling operations.

Two-currency arbitrage with capital controls

What happens if capital controls are imposed in one financial centre. If the transfer of capital is not allowed for financial transactions then arbitrage is not possible, and the divergence between the exchange rates in the two financial centres will persist. Nothing will happen to shift the supply and demand curves as in Figure 1.1. However, if capital controls are partial, the amount of capital allowed to be transferred from one financial centre to another may be inadequate to shift the supply and demand curves to the extent necessary to eliminate the discrepancy between the exchange rates. In this case, the

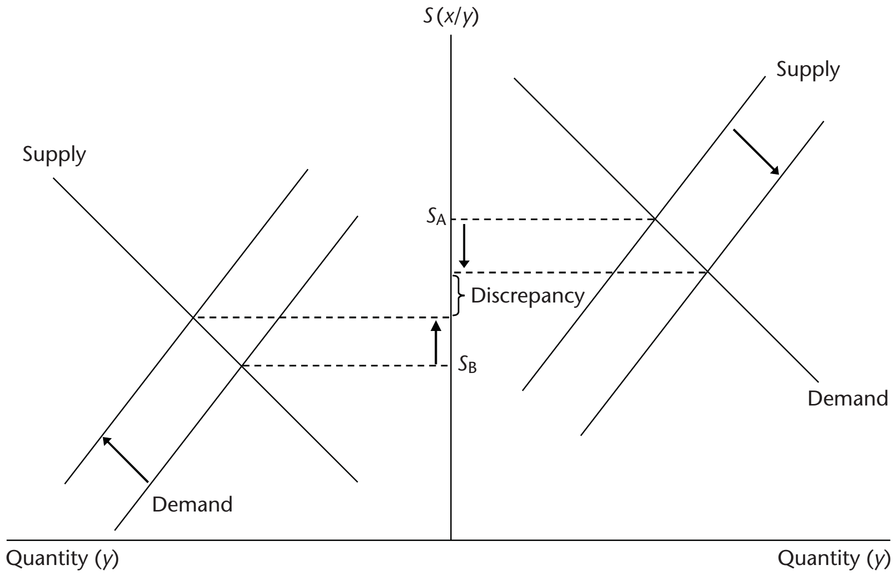

FIGURE 1.6 The effect of two-currency arbitrage in the presence of partial capital controls.

exchange rates in the two financial centres will approach each other, but they will not be equal. This situation is explained in Figure 1.6.

Two-currency arbitrage in the presence of the bid–offer spread

Let $S_{b,A}$ and $S_{b,B}$ be the bid rates, and $S_{a,A}$ and $S_{a,B}$ the offer rates in financial centres A and B respectively (still measured as $S(x/y)$). In the presence of the bid–offer spread, arbitragers buy (from market makers) at the higher offer rate and sell (to market makers) at the lower bid rate (note that $S_b < S_a$). Thus, the bid rate is determined by the demand of market makers and the supply of arbitragers. Conversely, the offer rate is determined by the demand of arbitragers and the supply of market makers. The bid and offer rates are related by the equation

$$S_a = S_b(1 + m) \tag{1.18}$$

where m is the bid–offer spread expressed as a percentage of the bid rate. For simplicity, we will assume that the bid–offer spread in financial centre A is equal to that prevailing in financial centre B.

Suppose now that the arbitrager wants to buy y in B and sell it in A. In this case arbitrage will be profitable if

$$\pi = S_{b,A} - S_{a,B} > 0 \tag{1.19}$$

which means that the no-arbitrage condition is given by

$$S_{b,A} = S_{a,B} \tag{1.20}$$

This situation is illustrated in Figure 1.7. Arbitragers buy y in B at $S_{a,B}$ and sell it in A at $S_{b,A}$. The process leads to a shift in the arbitragers' demand curve in B, causing a rise in $S_{a,B}$ and to a shift in the arbitragers' supply curve in A, causing a fall in $S_{b,A}$. The process continues until the two rates are equal. If only these changes take place, the bid–offer spread must rise in both A and B, and there is no reason why this should happen. In order that the spread stays at the same level, $S_{b,B}$ must rise and $S_{a,A}$ must fall. The following line of reasoning explains why this could take place. As $S_{a,B}$ rises, market makers find it profitable to increase the supply of y. To do this, they must obtain larger quantities of y by

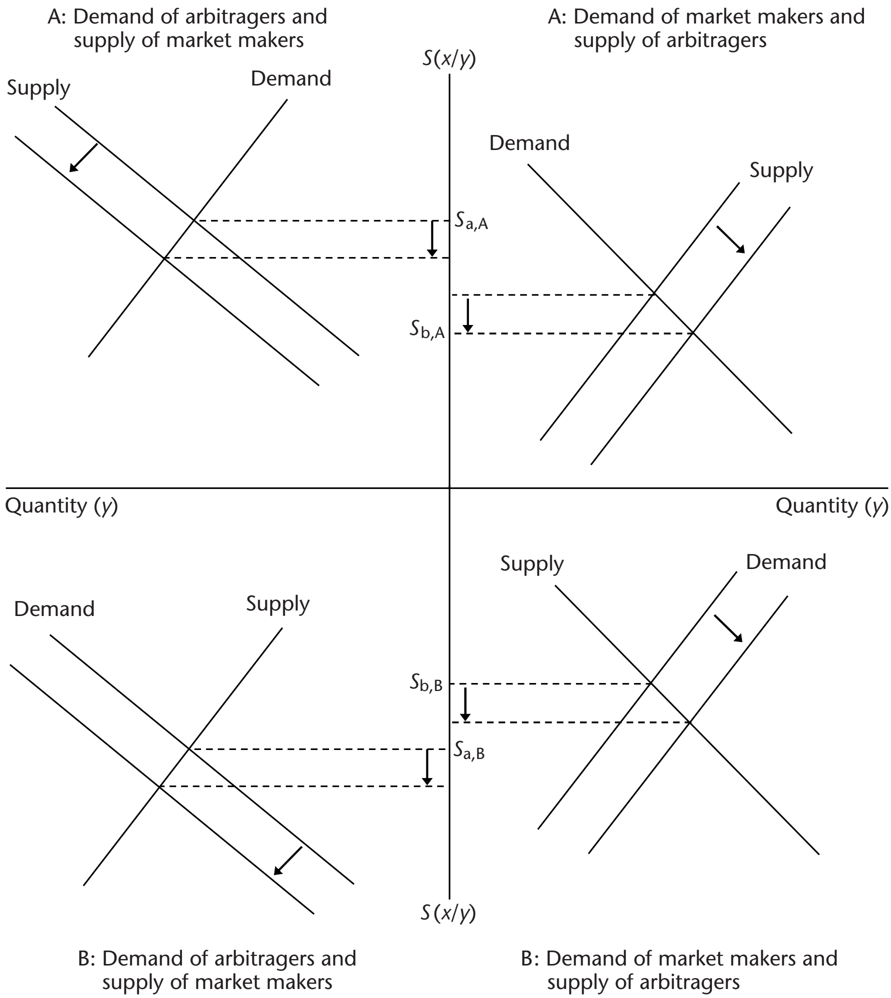

FIGURE 1.7 The effect of two-currency arbitrage in the presence of bid–offer spread.

buying it from customers. Thus the market makers' demand curve shifts, leading to an increase in $S_{b,B}$. Similarly, as $S_{b,A}$ declines, the market maker finds it cheaper to buy y, and the supply curve will shift to the left, leading to a fall in $S_{a,A}$.

Let us now consider the no-arbitrage condition in the presence of the bid–offer spread. Figure 1.8 shows a four-quadrant diagram, in which quadrants 1 and 3 show the no-arbitrage condition, whereas quadrants 2 and 4 show the relationship between the bid and offer rates. Notice that the line representing the relationship between the bid and offer rates (passing through the second and fourth quadrants) is less steep than the 45° line because the bid rate is always lower than the offer rate. The first quadrant shows the no-arbitrage line represented by equation (1.20). Any point above

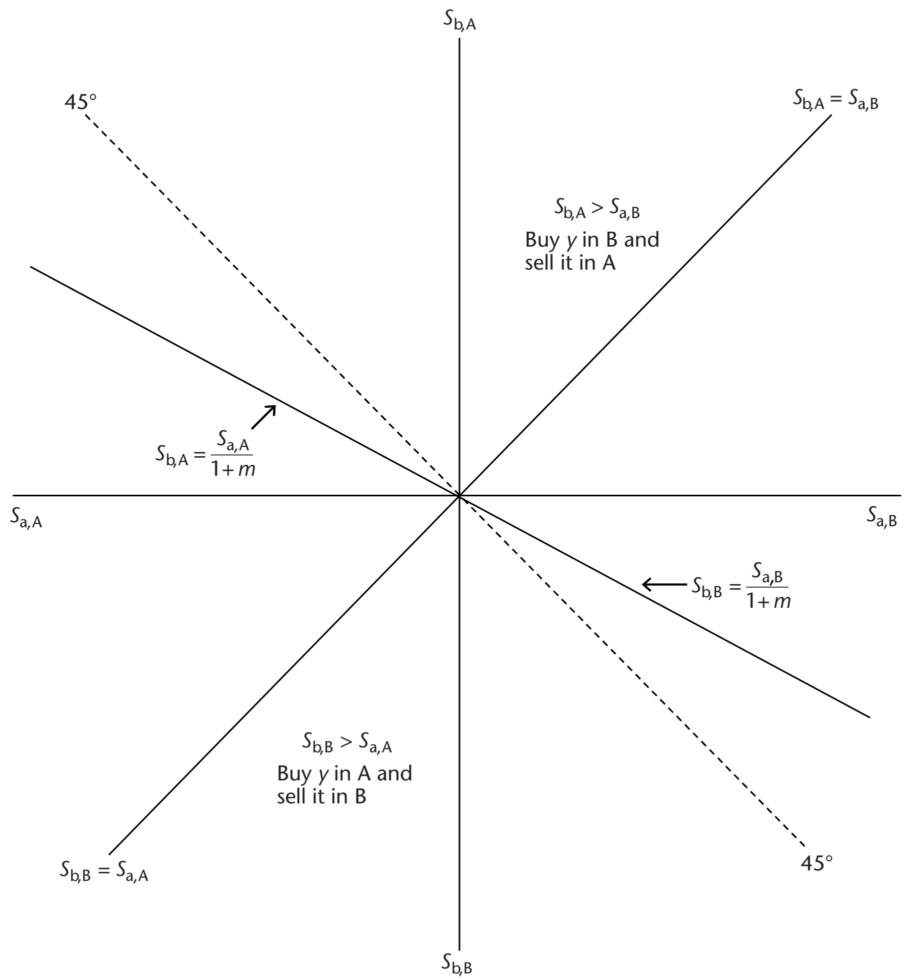

FIGURE 1.8 The no-arbitrage condition in the presence of bid–offer spread.

the line implies profitable arbitrage, with profit given by equation (1.19). In the third quadrant, points below the no-arbitrage line represent profitable arbitrage opportunities taking the form of buying y in A and selling it in B. In this case the profit is

$$\pi = S_{b,B} - S_{a,A} \tag{1.21}$$

The effect of the bid–offer spread is to reduce the profitability of arbitrage, since the spread is a transaction cost. Recall that equation (1.3) defines the arbitrage profit as the difference between the exchange rate in A (the sell rate) and the exchange rate in B (the buy rate). These rates were not defined as bid or offer rates, so let us assume that they are the mid-rates, which means

$$S_A = \tfrac{1}{2}[S_{b,A} + S_{a,A}] \tag{1.22}$$

$$S_B = \tfrac{1}{2}[S_{b,B} + S_{a,B}] \tag{1.23}$$

Arbitrage profit in the absence and presence of the bid–offer spread is given by equations (1.3) and (1.19) respectively. Since by definition

$$S_{b,A} < S_A \tag{1.24}$$

and

$$S_{a,B} > S_B \tag{1.25}$$

it follows that

$$S_{b,A} - S_{a,B} < S_A - S_B \tag{1.26}$$

which means that the presence of the bid–offer spread reduces the profitability of arbitrage, because the arbitrager has to buy at a higher rate and sell at a lower rate than otherwise.

Putting things together

Let us now consider the profitability of two-currency arbitrage in the presence of (i) bid–offer spread, (ii) fixed brokerage fees and (iii) Tobin tax. Consider the situation when the arbitrager buys y in B and sells it in A (equations 1.19 and 1.20). In the presence of fixed brokerage fees, β_A and β_B, and a Tobin tax, τ, which is assumed to be equal in both financial centres, arbitrage profit is reduced to

$$\begin{aligned}\pi &= S_{b,A}(1-\tau) - \beta_A - S_{a,B}(1+\tau) - \beta_B \\ &= [S_{b,A} - S_{a,B}](1-\tau) - (\beta_A + \beta_B)\end{aligned} \tag{1.27}$$

which means that profit is reduced further (that is, on top of the reduction resulting from the bid–offer spread). Equation (1.27) implies a no-arbitrage condition that is expressed as

$$S_{b,A} - S_{a,B} = \frac{\beta_A + \beta_B}{1-\tau} \tag{1.28}$$

which means that, for profitable two-currency arbitrage, the gap between the bid rate in A and the offer rate in B must be greater than $(\beta_A + \beta_B)/(1-\tau)$.

1.3 THREE-CURRENCY ARBITRAGE

Three-currency arbitrage, also known as triangular arbitrage and three-point arbitrage, works as follows. Given three currencies (x, y and z), three possible exchange rates exist: $S(x/y)$, $S(x/z)$ and $S(y/z)$. Since we are in this case dealing with three exchange rates, we will resort to the original exchange rate notation, which shows the units of measurement, as above. We say that the three exchange rates are consistent if

$$S(x/y) = \frac{S(x/z)}{S(y/z)} \tag{1.29}$$

Now, let us see what happens if an arbitrager tries to make profit by moving from one currency to another, ending up with the first currency. If the arbitrager ends up with one unit of the currency he or she started with, then arbitrage profit will be made. In general, if the condition (1.29) is violated then arbitrage profit can be made by moving in a particular direction and a loss will be made by moving in the opposite direction.

So, let us start with one unit of currency x, conducting arbitrage in the following manner:

1. Selling x and buying y to obtain $1/[S(x/y)]$ units of y.
2. Selling y and buying z to obtain $1/[S(x/y)S(y/z)]$ units of z.
3. Selling z and buying x to obtain $S(x/z)/[S(x/y)S(y/z)]$ units of x.

The profit realised from this operation (measured in units of x) is given by

$$\pi = \frac{S(x/z)}{S(x/y)S(y/z)} - 1 \tag{1.30}$$

If the condition represented by (1.29) is valid, it follows that $\pi = 0$, which means that (1.29) is the no-arbitrage condition. However, if

$$S(x/y) < \frac{S(x/z)}{S(y/z)} \tag{1.31}$$

it follows that

$$S(x/z) > S(x/y)S(y/z) \tag{1.32}$$

which means that $\pi > 0$. Hence, if the no-arbitrage condition (1.29) is violated, such that (1.31) is valid, then three-currency arbitrage will be profitable by the following sequence: $x \to y \to z \to x$.

Now, let us see what happens if the arbitrager follows the sequence $x \to y \to z \to x$, starting with one unit of x. This operation consists of the following steps

1. Selling x and buying z to obtain $1/[S(x/z)]$ units of z.
2. Selling z and buying y to obtain $S(y/z)/[S(x/z)]$ units of y.
3. Selling y and buying x to obtain $S(y/z)S(x/y)/[S(x/z)]$ units of x.

The profit realised from this operation is given by

$$\pi = \frac{S(y/z)S(x/y)}{S(x/z)} - 1 \tag{1.33}$$

Again, it is obvious that if (1.29) is valid then $\pi = 0$. In this case, profitable arbitrage is indicated by the violation of (1.29) such that

$$S(x/y) > \frac{S(x/z)}{S(y/z)} \tag{1.34}$$

because (1.34) implies that

$$S(y/z)S(x/y) > S(x/z) \tag{1.35}$$

which means that $\pi > 0$.

Just like two-currency arbitrage, three-currency arbitrage leads to a restoration of the no-arbitrage condition via changes in the supply of and demand for the three currencies. Let us trace what happens in the first case, as represented by (1.31). With the aid of Figure 1.9, we can see that each of the three steps results in changes in the forces of supply and demand as follows:

1. An increase in the demand for y (the supply of x), so $S(x/y)$ rises.
2. An increase in the demand for z (the supply of y), so $S(y/z)$ rises.
3. An increase in the demand for x (the supply of z), so $S(x/z)$ falls.

These changes in supply and demand will restore the equilibrium condition.

Three-currency arbitrage in the presence of bid–offer spreads

Let us see what happens if an arbitrager wants to follow the sequence $x \to z \to y \to x$ in the presence of bid–offer spreads. The operation consists of the following steps:

1. Buying z against x at $S_a(x/z)$ to obtain $1/[S_a(x/z)]$ units of z.
2. Buying y against z at $S_b(y/z)$ to obtain $S_b(y/z)/[S_a(x/z)]$ units of y.
3. Buying x against y at $S_a(x/y)$ to obtain $S_b(y/z)S_a(x/y)/[S_a(x/z)]$ units of x.

For this operation to be profitable, the following condition must be satisfied:

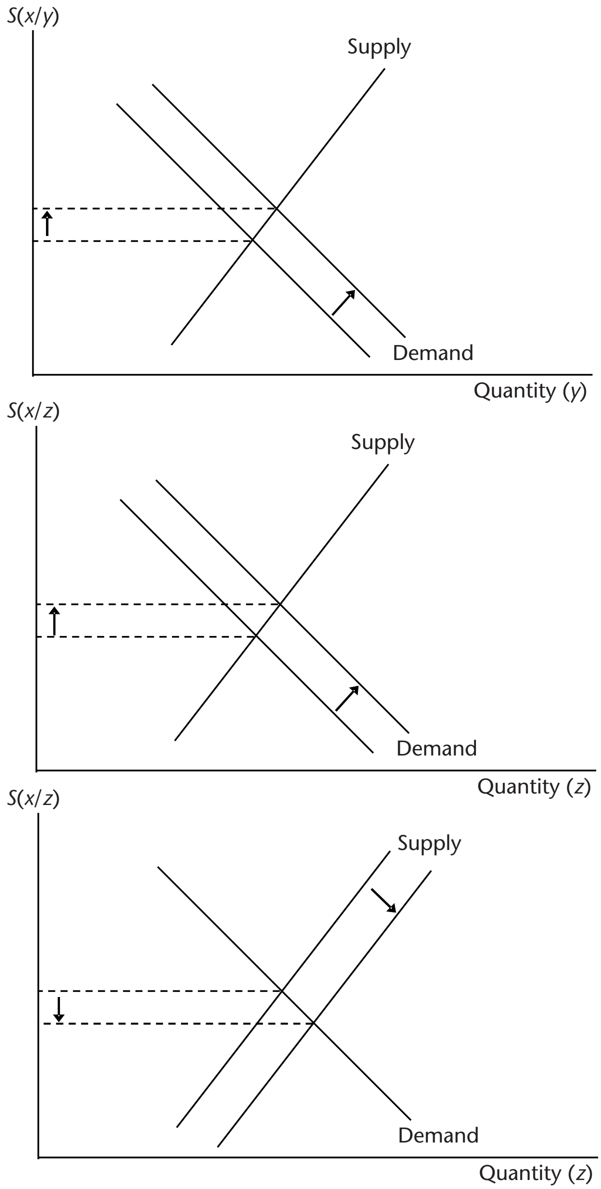

FIGURE 1.9 The effect of three-currency arbitrage.

$$\frac{S_b(y/z)S_a(x/y)}{S_a(x/z)} > 1 \tag{1.36}$$

which gives

$$\pi = \frac{S_b(y/z)S_a(x/y)}{S_a(x/z)} - 1 \tag{1.37}$$

in which case, the no-arbitrage condition is

$$S_a(x/y) = \frac{S_a(x/z)}{S_b(y/z)} \tag{1.38}$$

Likewise, it can be shown that the sequence $x \to y \to z \to x$ can be profitable if

$$\frac{S_b(x/z)}{S_b(x/y)S_a(y/z)} > 1 \tag{1.39}$$

which gives

$$\pi = \frac{S_b(x/z)}{S_b(x/y)S_a(y/z)} - 1 \tag{1.40}$$

in which case, the no-arbitrage condition is

$$S_b(x/y) = \frac{S_b(x/z)}{S_a(y/z)} \tag{1.41}$$

Equations (1.38) and (1.41) are used to calculate the bid and offer cross exchange rates when currency z is the numeraire.

1.4 MULTI-CURRENCY ARBITRAGE

Consider arbitrage involving four currencies: x_1, x_2, x_3 and x_4 by following the sequence $x_1 \to x_2 \to x_3 \to x_4 \to x_1$. Arbitrage consists of the following steps:

1. Buying x_2 and selling x_1 at $S(x_1/x_2)$ to obtain $1/[S(x_1/x_2)]$ units of x_2.
2. Buying x_3 and selling x_2 at $S(x_2/x_3)$ to obtain $1/[S(x_1/x_2)S(x_2/x_3)]$ units of x_3.
3. Buying x_4 and selling x_3 at $S(x_3/x_4)$ to obtain $1/[S(x_1/x_2)S(x_2/x_3)S(x_3/x_4)]$ units of x_4.
4. Buying x_1 at $S(x_1/x_4)$ to obtain $S(x_1/x_4)/[S(x_1/x_2)S(x_2/x_3)S(x_3/x_4)]$ units of x_1.

This operation will be profitable if

$$\pi = \frac{S(x_1/x_4)}{S(x_1/x_2)S(x_2/x_3)S(x_3/x_4)} - 1 > 0 \tag{1.42}$$

in which case the no-arbitrage condition is

$$S(x_1/x_4) = S(x_1/x_2)S(x_2/x_3)S(x_3/x_4) \tag{1.43}$$

or

$$S(x_1/x_2)S(x_2/x_3)S(x_3/x_4)S(x_4/x_1)=1 \quad (1.44)$$

In general, an n-currency arbitrage is profitable if the following no-arbitrage condition is violated:

$$S(x_1/x_2)S(x_2/x_3)S(x_3/x_4)...S(x_{n-1}/x_n)S(x_n/x_1) = 1 \quad (1.45)$$

which means that even two-currency arbitrage can be represented as a special case of (1.45). If $n = 2$, the no-arbitrage condition reduces to

$$S(x_1/x_2)S(x_2/x_1) = 1 \quad (1.46)$$

Chacholiades (1971) has shown that if three-currency arbitrage is not profitable, then n-currency arbitrage is not profitable either. This means that for equation (1.45) to be satisfied, a necessary and sufficient condition is

$$S(x_1/x_2)S(x_2/x_3)S(x_3/x_1) = 1 \quad (1.47)$$

The proof of this proposition is based on mathematical induction. If (n-1)-currency arbitrage is not profitable, then n-currency arbitrage is not profitable either. For unprofitable n-currency arbitrage, equation (1.45) must hold. Since (n-1)-currency arbitrage is not profitable by assumption, the following equation must be satisfied

$$S(x_1/x_2)S(x_2/x_3)S(x_3/x_4)...S(x_{n-2}/x_{n-1})S(x_{n-1}/x_1) = 1 \quad (1.48)$$

Dividing (1.44) by (1.48) we obtain

$$\frac{S(x_{n-1}/x_n)S(x_n/x_1)}{S(x_{n-1}/x_1)}=1 \quad (1.49)$$

which, for $n = 3$, is equivalent to (1.47) because $S(x_1/x_2) = 1/[S(x_2/x_1)]$. Hence, if (1.47) and (1.48) are satisfied, (1.45) must also be satisfied, which proves the proposition.

Multi-currency arbitrage with bid–offer spreads

In the presence of bid–offer spreads, the operation takes the following form:

1. Buying x_2 and selling x_1 at $S_a(x_1/x_2)$ to obtain $1/[S_a(x_1/x_2)]$ units of x_2.
2. Buying x_3 and selling x_2 at $S_a(x_2/x_3)$ to obtain $1/[S_a(x_1/x_2)S_a(x_2/x_3)]$ units of x_3.
3. Buying x_4 at $S_a(x_3/x_4)$ to obtain $1/[S_a(x_1/x_2)S_a(x_2/x_3)S_a(x_3/x_4)]$ units of x_4.
4. Buying x_1 at $S_b(x_1/x_4)$ to obtain $S_b(x_1/x_4)/[S_a(x_1/x_2)S_a(x_2/x_3)S_a(x_3/x_4)]$ units of x_1.

This operation will be profitable if

$$\pi = \frac{S_b(x_1/x_4)}{S_a(x_1/x_2)S_a(x_2/x_3)S_a(x_3/x_4)} - 1 > 0 \quad (1.50)$$

in which case the no-arbitrage condition is

$$S_b(x_1/x_4) = S_a(x_1/x_2)S_a(x_2/x_3)S_a(x_3/x_4) \tag{1.51}$$

or

$$S_a(x_1/x_2)S_a(x_2/x_3)S_a(x_3/x_4)S_a(x_4/x_1) = 1 \tag{1.52}$$

because $S_b(x_1/x_4) = 1/[S_a(x_4/x_1)]$. If the bid–offer spread is the same for all exchange rates, the condition becomes

$$S_b(x_1/x_2)S_b(x_2/x_3)S_b(x_3/x_4)S_b(x_1/x_4)(1 + m)^4 = 1 \tag{1.53}$$

Hence the n-currency no-arbitrage condition in the presence of bid–offer spread is given by

$$S_a(x_1/x_2)S_a(x_2/x_3)S_a(x_3/x_4)...S_a(x_n/x_1) = 1 \tag{1.54}$$

or

$$S_b(x_1/x_2)S_b(x_2/x_3)S_b(x_3/x_4)...S_b(x_n/x_1)(1 + m)^n = 1 \tag{1.55}$$

1.5 EXAMPLES

Table 1.1 reports some bilateral exchange rates as on 16 December 2001. We can use these figures to check whether or not the no-arbitrage conditions associated with three-currency, four-currency and five-currency arbitrage are valid.

Table 1.2 lists possible sequences for three-currency, four-currency and five-currency arbitrage, the associated conditions and whether or not the conditions are satisfied. The numbers appearing in the third column are the products of the exchange rates as implied by the general no-arbitrage condition (1.45). The no-arbitrage condition will be satisfied if the product is 1, indicating zero profit. This is because unity signifies that the no-arbitrage condition implies that when the arbitrager starts with one unit of a particular

TABLE 1.1 Exchange rates on 16 December 2001.

x/y	USD	SEK	DKK	NZD	EUR	AUD
USD	1					
SEK	10.54	1				
DKK	8.2449	0.7819	1			
NZD	2.3941	0.2271	0.2904	1		
EUR	1.1073	0.1050	0.1343	0.4625	1	
AUD	1.9296	0.1830	0.2340	0.8060	1.7246	1

Source: Bloomberg.

TABLE 1.2 Examples of *n*-currency arbitrage.

Arbitrage	Sequence	Condition
Three-currency	SEK → USD → NZD → SEK	0.9998
Three-currency	AUD → EUR → DKK → AUD	0.9898
Four-currency	USD → NZD → DKK → SEK → USD	0.9997
Four-currency	EUR → NZD → SEK → AUD → EUR	0.9898
Five-currency	SEK → USD → DKK → NZD → EUR → SEK	0.9994
Five-currency	AUD → EUR → NZD → SKK → DKK → AUD	0.9900

currency, she ends up with one unit of the same currency. The calculation of the figures in the third column can be illustrated by reference to the first arbitrage operation. In this case we have

$$\begin{aligned} S(\text{SEK/USD}) \times S(\text{USD/NZD}) \times S(\text{NZD/SEK}) &= 10.5400 \times \frac{1}{2.3941} \times 0.2271 \\ &= 0.9998 \end{aligned}$$

We can see that all of the numbers are close to one, implying the absence of profitable arbitrage operations if we assume that the slight difference between unity and the figures shown in the table is due to rounding. If it is not due to rounding, then there is still no possibility for profitable arbitrage because the difference is so small that it is bound to be consumed by transaction costs.

Let us now assume that there is a 0.1% bid–offer spread in all exchange rates, such that we have the following information:

$$S(\text{SEK/USD}) = 10.5295 - 10.5505$$

$$S(\text{NZD/USD}) = 2.3917 - 2.3965$$

$$S(\text{NZD/SEK}) = 0.2269 - 0.2273$$

In this case, the no-arbitrage condition is checked as follows

$$\begin{aligned} S_a(\text{SEK/USD}) \times S_a(\text{USD/NZD}) \times S_a(\text{NZD/SEK}) &= 10.5505 \times \frac{1}{2.3965} \times 0.2273 \\ &= 1.0008 \end{aligned}$$

which is again close to unity, implying the absence of profitable arbitrage.

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