# Expressions and equations

**1** Show that any flow of heat between two heat reservoirs $A$ and $B$ at temperatures $T_A$ and $T_B$ respectively must be from the hotter to the cooler.

Working

**2** A mass moves through the air and the resistance it experiences is found to be directly proportional to the velocity of the mass. Show that if a mass falls towards the earth then, if the positive direction is taken as downwards, its velocity $v(t)$ at time $t$ satisfies the differential equation:

$$m\frac{\mathrm{d}v(t)}{\mathrm{d}t} = mg - kv(t)$$

where $k$ is a positive constant and $g$ is the acceleration due to gravity.

Working

**3** In an emergency airlift of supplies, a parcel of tents of total mass $m$ is dropped from a plane flying at a great height and as it falls through the air the parcel experiences an air resistance that is proportional to the square of its velocity. Show that the equation that describes this state of affairs is:

$$m\frac{\mathrm{d}v(t)}{\mathrm{d}t} = mg - kv^2(t)$$

where $v(t)$ is its velocity at time $t$ and $g$ is the acceleration due to gravity. Show further that if the direction down is taken to be positive, the terminal velocity of the parcel is given as:

$$\sqrt{\frac{mg}{k}}$$

Working

**4** The sun radiates energy and heats up the earth. At the same time the earth emits radiation that it receives from the sun. The total power $P_s$ (energy per second) radiated by the sun (black body radiation) and the total power $P_e$ absorbed by the earth are given as:

$$P_s = 4\pi\sigma R_s{}^2 T_s{}^4, \quad P_e = \frac{P_s R_e{}^2}{4d^2}$$

where $R_s$ and $R_e$ are the radii of the sun and the earth, $T_s$ and $T_e$ the surface temperatures of the sun and the earth, $d$ is the distance between the sun and the earth and $\sigma$ is the Stefan-Boltzman constant. Assuming that the earth is in thermal equilibrium with itself, that is, the power absorbed is equal to the power emitted, show that the surface temperature of the sun is given as:

$$T_s = T_e \sqrt[4]{\frac{4d^2}{R_s{}^2}}$$

Working

## Working

**1** When a finite quantity of heat $\Delta Q$ is added to or extracted from a heat reservoir at constant temperature $T$ it undergoes a finite entropy change $\Delta S$ where:

$$\Delta S = \frac{\Delta Q}{T}$$

Therefore, the entropy change of reservoir $A$ is:

$$\Delta S_A = \frac{\Delta A_A}{T_A}$$

and the entropy change for reservoir $B$ is:

$$\Delta S_B = \frac{\Delta B_A}{T_B}$$

The total entropy change for the two reservoir system is given as:

$$\Delta S_{total} = \Delta S_A + \Delta S_B$$

where $\Delta S_{total} > 0$ by the second law of thermodynamics.

Therefore

$$\Delta S_{total} = \Delta S_A + \Delta S_B = \frac{\Delta Q_A}{T_A} + \frac{\Delta Q_B}{T_B} > 0$$

If the quantity of heat $\Delta Q$ passes from one reservoir to another the heat lost from one is considered as negative and the amount gained by the other is considered as positive so that:

$$|\Delta Q_B| = |\Delta Q_A| \quad \text{but} \quad \Delta Q_B = -\Delta Q_A$$

Substituting in the above equation for the total entropy of the two reservoir system we see that:

$$\begin{aligned}
\Delta S_{\text{total}} &= \frac{\Delta Q_A}{T_A} + \frac{\Delta Q_B}{T_B} \\
&= \frac{\Delta Q_A}{T_A} - \frac{\Delta Q_A}{T_B} \\
&= \Delta Q_A\left(\frac{1}{T_A} - \frac{1}{T_B}\right) \\
&= \Delta Q_A\left(\frac{T_B - T_A}{T_A T_B}\right) > 0
\end{aligned}$$

Therefore either $\Delta Q_A > 0$ and $T_B > T_A$ or $\Delta Q_A < 0$ and $T_B < T_A$ This means that if reservoir $A$ gains heat then it is cooler than reservoir $B$ or if reservoir $A$ loses heat then it is hotter than reservoir $B$. In either case heat flows from the hotter to the cooler reservoir.

Questions

**2** As the mass falls it accelerates downwards. This is caused by a downwards force due to gravity and a resistive force acting upwards. The net force is downwards giving rise to a net acceleration. That is:

$$m\frac{dv(t)}{dt} = mg - kv(t)$$

where the positive direction is taken as downwards.

Questions

**3** As the parcel falls it accelerates downwards. This is caused by a downwards force due to gravity and a resistive force acting upwards. The net force is downwards giving rise to a net acceleration. That is:

$$m\frac{dv(t)}{dt} = mg - kv^2(t)$$

Eventually, the resistive force will be sufficient to overcome the downward gravitational force thereby causing to body to have zero acceleration. That is:

$$m\frac{dv(t)}{dt} = mg - kv^2(t) = 0 \quad \text{and so} \quad mg = kv^2(t) \quad \text{that is} \quad v(t) = \sqrt{\frac{mg}{k}}$$

Questions

**4** Assuming the earth radiates according to the same mechanisms as the sun (black body) then we can say that:

$$\begin{aligned} P_e &= 4\pi\sigma {R_e}^2 {T_e}^4 && \text{by analogy with } P_s = 4\pi\sigma {R_s}^2 {T_s}^4 \\ &= \frac{P_s {R_e}^2}{4d^2} && \text{as given} \\ &= \frac{4\pi\sigma {R_s}^2 {T_s}^4 {R_e}^2}{4d^2} && \text{substituting for } P_s \\ &= \frac{\pi\sigma {R_s}^2 {T_s}^4 {R_e}^2}{d^2} && \text{by cancellation} \end{aligned}$$

and so:

$$\frac{\pi\sigma {R_s}^2 {T_s}^4 {R_e}^2}{d^2} = 4\pi\sigma {R_e}^2 {T_e}^4$$

that is:

$$\frac{{R_s}^2 {T_s}^4}{d^2} = 4{T_e}^4 \qquad \text{by cancelling common factors on each side}$$

therefore:

$${T_s}^4 = \frac{{T_e}^4 4d^2}{{R_s}^2} \quad \text{and so} \quad T_s = T_e \sqrt[4]{\frac{4d^2}{{R_s}^2}}$$

Questions