Graphs

1 A large population of insects is introduced into a closed habitat. It is found that the number of insects, N at a time t days after the introduction is given by:

$$N = \frac{1500}{1 - 0{\cdot}7e^{-0{\cdot}03t}}$$

(a) How many insects were originally introduced?
(b) Find the value of t when $N = 1900$ by solving a suitable equation.
(c) Sketch the graph of N against t for $0 \leq t \leq 100$.
(d) What is the ultimate value of N?

Solutions Working

2 The number of defects in a trial manufacturing process were recorded in the following table:

No of hours	1	2	3	4	5	6	7
No of defects	20	30	52	77	135	165	180

It is thought that the number of defects n found after t hours is given approximately by an equation of the form:

$$n = ae^{kt}$$

where a and k are constants.

(a) Explain why, if this equation is true, the graph of $\ln n$ against t will be a straight line.
(b) Plot the graph of $\ln n$ against t for this data and show that the first five points lie approximately on a straight line. Hence find the values of the constants a and k.

Solutions Working

3 How many pieces of cable of lengths 245 cm and 84 cm can be cut from a 3000 cm continuous length of cable so as to minimise wastage?

Solutions Working

4 Two machines A and B operating concurrently provide outputs $A(t)$ and $B(t)$ respectively where t is a time parameter. Plot the graphs of $A(t)$ and $B(t)$ on the same set of axes using your spreadsheet to find a first approximation of when their outputs coincide where:

(a) $A(t) = 20t - 5$ and $B(t) = t^2 - t + 1$ for $0 \le t \le 5$ in steps of $t = 0{\cdot}2$
(b) $A(t) = t^2 - 2t + 3$ and $B(t) = 15 + t - 4t^2$ for $0 \le t \le 5$ in steps of $t = 0{\cdot}2$
(c) $A(t) = 25\cos t$ and $B(t) = e^t$ for $0 \le t \le 5$ with t in radians in steps of $t = 0{\cdot}2$
(d) $A(t) = 15\cos t$ and $B(t) = t^3 - 2t^2 - t + 2$ for $0 \le t \le 4$ in steps of $t = 0{\cdot}2$

Solutions Working

Solutions

1
(a) 5000
(b) 40·05 days
(d) 1500

Questions Working

2
(b) $a \cong 12{\cdot}18$ to 2 dp and $k \cong 0{\cdot}48$

Questions Working

3 4 lengths of 245 cm and 24 lengths of 84 cm

Questions Working

4
(a) $t = 0{\cdot}3$
(b) $t = 1{\cdot}9$
(c) $t = 1{\cdot}4$
(d) $t = 1{\cdot}6$

Questions Working

Working

1
(a) We know that $N = \dfrac{1500}{1 - 0{\cdot}7e^{-0{\cdot}03t}}$ at time t.

When $t = 0$, $N = \dfrac{1500}{1 - 0{\cdot}7e^0} = \dfrac{1500}{1 - 0{\cdot}7} = 5000$ which is the number of insects originally introduced.

(b) To find the value of t when $N = 1900$ we must solve

$$1900 = \frac{1500}{1 - 0{\cdot}7e^{-0{\cdot}03t}} \quad \text{for } t.$$

We see that:

$$1900(1 - 0{\cdot}7^{-0{\cdot}03t}) = 1500$$

so that

$$1900 - 1900 \times 0{\cdot}7e^{-0{\cdot}03t} = 1500$$

so that

$$-1900 \times 0{\cdot}7e^{-0{\cdot}03t} = 1500 - 1900 = -400$$

so that

$$e^{-0{\cdot}03t} = \frac{400}{1900 \times 0{\cdot}7} = 0{\cdot}30075\ldots$$

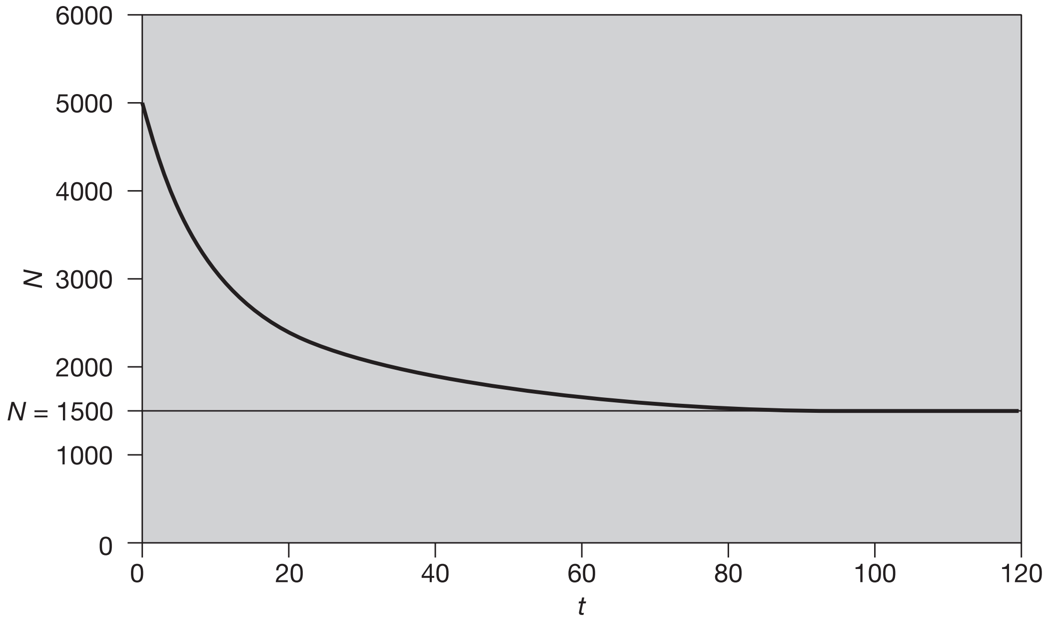
so that

$$-0{\cdot}03t = (0{\cdot}3008) = -1{\cdot}20146\ldots$$

so that

$$t = \frac{-1{\cdot}2015}{-0{\cdot}03} = 40{\cdot}05 \text{ days to 2 dp.}$$

(c)



(d) As $t \to \infty$ so $N \to 1500$ and this is the ultimate value.

Questions Solutions

2 (a) Given the equation $n = ae^{kt}$, taking logs of both sides yields:

$$\ln n = \ln\left(ae^{kt}\right) = \ln a + \ln e^{kt} = \ln a + kt \ln e = \ln a + kt \text{ since } \ln e = 1$$

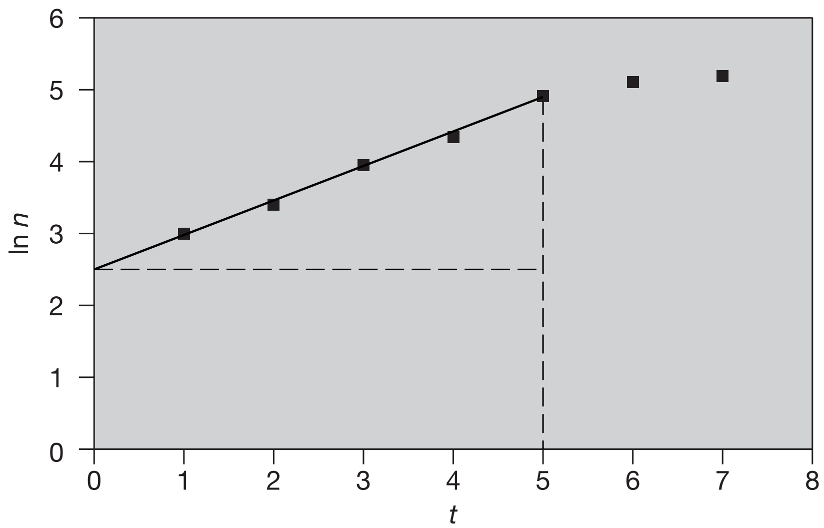
This equation is of the straight line form $y = mx + c$ where:

$y = \ln n$
$x = t$
$m = k$ – the gradient of the line
$c = \ln a$ – the vertical intercept

Therefore, plotting $\ln n$ against t will give a straight line with gradient k and vertical intercept $\ln a$.

(b)

No of hours	1	2	3	4	5	6	7
No of defects	20	30	52	77	135	165	180
$\ln n$	3·0	3·4	4·0	4·3	4·9	5·1	5·2



The first five points lie on a straight line which is extended to the vertical axis so as to enable us to find the equation of the line.

From the line of best fit drawn on the graph we see that the vertical intercept is at $\ln n = \ln a = 2{\cdot}5$ so the value of a is:

$a = e^{\ln a} \cong e^{2\cdot5} = 12{\cdot}18$ to 2 dp

and the gradient of the line of best fit is approximately:

$$k \cong \frac{4{\cdot}9 - 2{\cdot}5}{5} = 0{\cdot}48$$

The equation of the line of best fit is then:

$\ln n = 12{\cdot}18 + 0{\cdot}48t$ so that $n = 12{\cdot}18e^{0\cdot48t}$

Questions Solutions

3 Let x be the number of cables of length 245 cm and y the number of length 84 cm. The wastage is then:

$$w = 3000 - 245x - 84y$$

The maximum number of cables of length 245 cm that can be cut from the 3000 cm cable is 12 with an amount of wastage that is less than 84cm. So we construct a spreadsheet table of x-values ranging from 12 to 0 in column A. In column B we calculate the length of cable left over after the specified number of lengths of 245 cm have been cut off and then divide that number by 84 to see how many lengths of 84 cm can be cut from what is left. The formula we use is:

$$= (3000 - x * 245)/84$$ [the value of x is the appropriate cell address of the x-value]

This produces a number that is not a whole number so in column C we use the INT function to find the nearest whole number less than the number in the adjacent cell this is the y-value. In column D we use the formula:

$$= 3000 - 245 * x - 84 * y$$

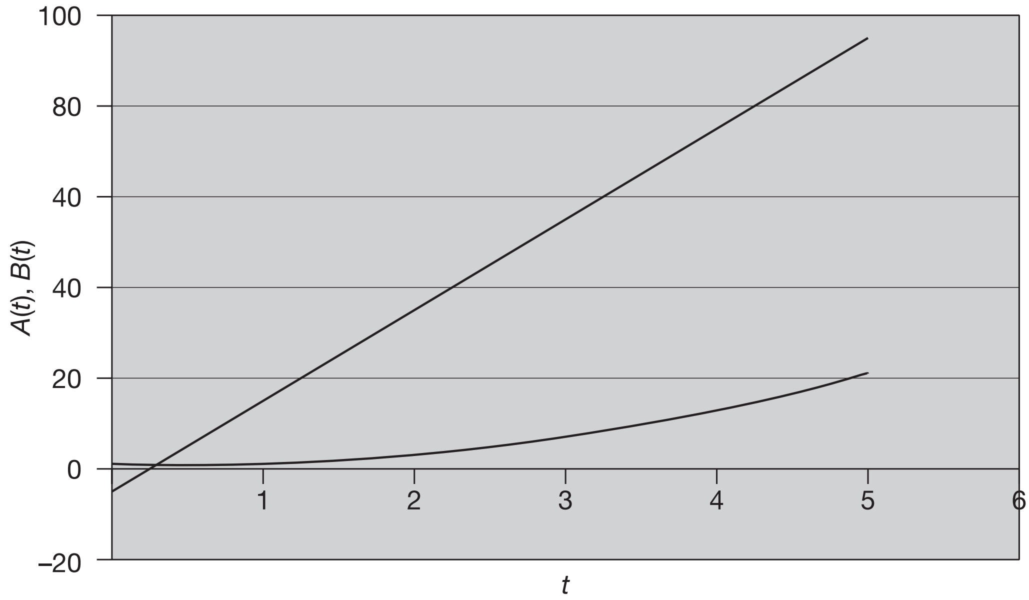
to calculate the wastage.

x		y	w
12	0·714286	0	60
11	3·630952	3	53
10	6·547619	6	46
9	9·464286	9	39
8	12·38095	12	32
7	15·29762	15	25
6	18·21429	18	18
5	21·13095	21	11
4	24·04762	24	4
3	26·96429	26	81
2	29·88095	29	74
1	32·79762	32	67
0	35·71429	35	60

Clearly the wastage is a minimum when we cut 4 lengths of 245 cm and 24 lengths of 84 cm.

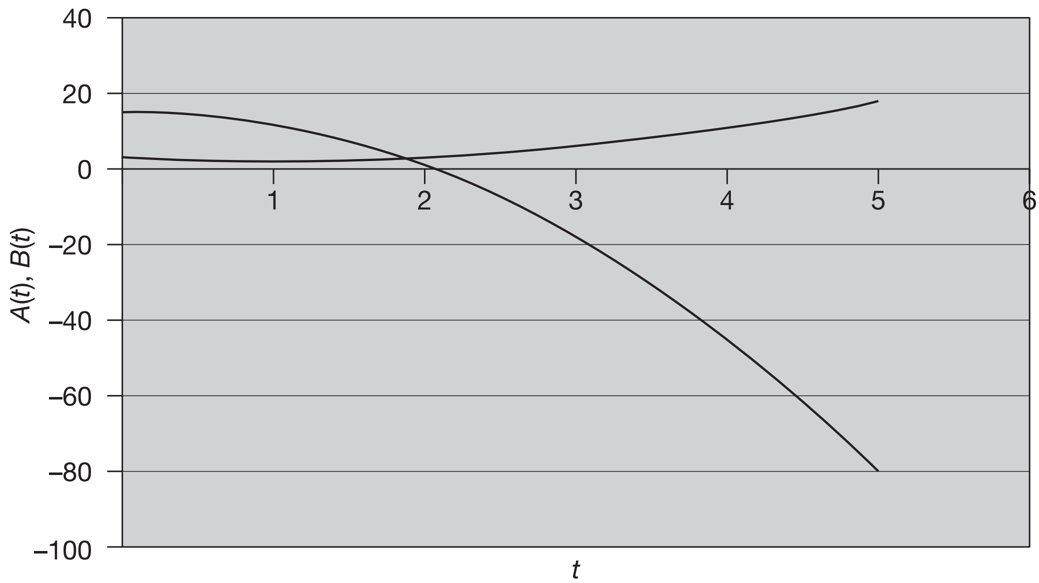
Questions Solutions

4 (a) $A(t) = 20t - 5$ and $B(t) = t^2 - t + 1$ for $0 \le t \le 5$ in steps of $t = 0{\cdot}2$



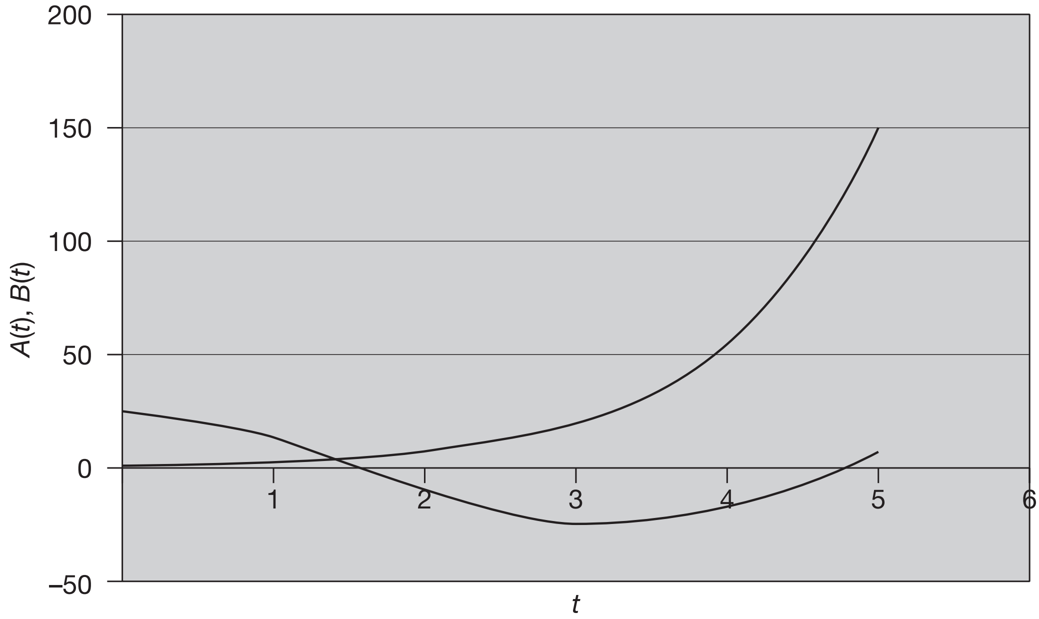
An approximate value for the point of intersection is $t = 0{\cdot}3$. Recall that two graphs can be plotted on the same graph provided there is a blank row between the two sets of data.

(b) $A(t) = t^2 - 2t + 3$ and $B(t) = 15 + t - 4t^2$ for $0 \le t \le 5$ in steps of $t = 0{\cdot}2$



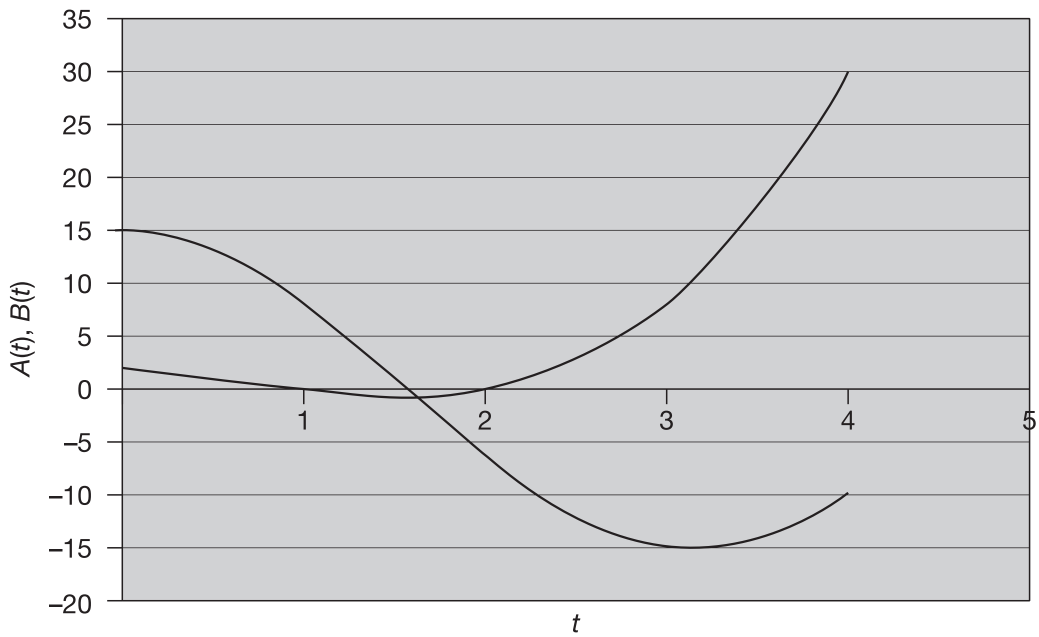
An approximate value for the point of intersection is $t = 1{\cdot}9$.

(c) $A(t) = 25\cos t$ and $B(t) = e^t$ for $0 \leq t \leq 5$ with t in radians in steps of $t = 0{\cdot}2$



An approximate value for the point of intersection is $t = 1{\cdot}4$.

(d) $A(t) = 15\cos t$ and $B(t) = t^3 - 2t^2 - t + 2$ for $0 \leq t \leq 4$ in steps of $t = 0{\cdot}2$



An approximate value for the point of intersection is $t = 1{\cdot}6$.

Questions Solutions