Partial fractions

1 The functional specification for the safety of a structure consisting of a network of cables is given as:

$$S(x) = \frac{23x - 11x^2}{(1-2x)(9-x^2)}$$

where x is a positive, non-zero stress parameter of the wire that is used to construct a cable and $S(x)$ is the parameterized safety factor of the structure. By separating into partial fractions show that:

$$S(x) = \frac{23x}{9} + \frac{105x^2}{27} + \frac{653x^3}{81} + \dots \quad \text{valid only for } 0 < x < 0{\cdot}5$$

[*Hint*: $(1-a)^{-1} = 1 + a + a^2 + a^3 + \dots$ and $(1+a)^{-1} = 1 - a + a^2 - a^3 + \dots$ provided $-1 < a < 1$]

Working

2 The transfer function for a linear, time-invariant control system is given as:

$$H(s) = \frac{1 - s^{-1}}{1 - 8s^{-3}}$$

Show that

$$H(s) = 1 + \frac{1}{3(s-2)} - \frac{10}{3\left([s+1]^2 + (\sqrt{3})^2\right)} - \frac{4s}{3\left([s+1]^2 + (\sqrt{3})^2\right)}$$

Working

Working

1 Assume that, since $9 - x^2 = (3+x)(3-x)$:

$$\frac{23x - 11x^2}{(1-2x)(9-x^2)} = \frac{A}{1-2x} + \frac{B}{3+x} + \frac{C}{3-x}$$
$$= \frac{A(3+x)(3-x) + B(1-2x)(3-x) + C(1-2x)(3+x)}{(1-2x)(3+x)(3-x)}$$

Then, equating numerators:

$$23x - 11x^2 \equiv A(3+x)(3-x) + B(1-2x)(3-x) + C(1-2x)(3+x)$$

Let $x = 3$ so that $69 - 99 = C(-5)(6)$ and so $C = 1$

Let $x = -3$ so that $-69 - 99 = B(7)(6)$ and so $B = -4$

Let $x = 1/2$ so that $23/2 - 11/4 = A(7/2)(5/2)$ and so $A = 1$, therefore:

$$\frac{23 - 11x^2}{(1-2x)(9-x^2)} = \frac{1}{1-2x} - \frac{4}{3+x} + \frac{1}{3-x}$$

Expanding each of the terms on the right-hand side we see that:

$$\frac{1}{(1-2x)} = \left(1 + 2x + 4x^2 + 8x^3 + \ldots\right)$$

valid for $-1 < 2x < 1$ that is $-1/2 < x < 1/2$

$$-\frac{4}{(3+x)} = -\frac{4}{3(1+x/3)} = -\frac{4}{3}\left(1 + -\frac{x}{3} + \frac{x^2}{9} + \frac{x^3}{27} + \ldots\right)$$

valid for $-1 < x/3 < 1$ that is $-3 < x < 3$

$$\frac{1}{(3-x)} = \frac{1}{3(1-x/3)} = \frac{1}{3}\left(1 + \frac{x}{3} + \frac{x^2}{9} + \frac{x^3}{27} + \ldots\right)$$

valid for $-1 < x/3 < 1$ that is $-3 < x < 3$

So that:

$$\frac{1}{(1-2x)} - \frac{4}{(3+x)} + \frac{1}{(3-x)}$$
$$= \left(1 + 2x + 4x^2 + 8x^3 + \ldots\right) - \frac{4}{3}\left(1 - \frac{x}{3} + \frac{x^2}{9} - \frac{x^3}{27} + \ldots\right)$$
$$+ \frac{1}{3}\left(1 + \frac{x}{3} + \frac{x^2}{9} + \frac{x^3}{27} + \ldots\right)$$
$$= 1 - \frac{4}{3} + \frac{1}{3} + x\left(2 + \frac{4}{9} + \frac{1}{9}\right) + x^2\left(4 - \frac{4}{27} + \frac{1}{27}\right) + x^3\left(8 + \frac{4}{81} + \frac{1}{81}\right) + \ldots$$
$$= \frac{23x}{9} + \frac{105x^2}{27} + \frac{653x^3}{81} + \ldots$$

This is valid for $0 < x < 1/2$ since x is stated as being a positive, non-zero number.

Questions

2

$$H(s) = \frac{1 - s^{-1}}{1 - 8s^{-3}}$$

$$= \frac{s^3 - s^2}{s^3 - 8}$$ obtained by multiplying numerator and denominator by s^3

$$= 1 + \frac{8 - s^2}{s^3 - 8}$$ after division

Now:

$$\frac{8 - s^2}{s^3 - 8} = \frac{8 - s^2}{(s - 2)(s^2 + 2s + 4)}$$

$$= \frac{A}{s - 2} + \frac{Bs + C}{s^2 + 2s + 4}$$ breaking into partial fractions

$$= \frac{A(s^2 + 2s + 4) + (Bs + C)(s - 2)}{(s - 2)(s^2 + 2s + 4)}$$

Equating numerators:

$$8 - s^2 \equiv A(s^2 + 2s + 4) + (Bs + C)(s - 2)$$

Let $s = 2$ so that $8 - 4 = A(4 + 4 + 4)$ and so $A = 1/3$
Let $s = 0$ so that $8 = 4A - 2C$ and so $C = 2A - 4 = -10/3$
Let $s = 1$ so that $8 - 1 = 7A - (B + C)$ and so $B = 7A - C - 7 = -4/3$

Therefore:

$$H(s) = 1 + \frac{1}{3(s - 2)} - \frac{4s + 10}{3(s^2 + 2s + 4)}$$

$$= 1 + \frac{1}{3(s - 2)} - \frac{10}{3(s^2 + 2s + 4)} - \frac{4s}{3(s^2 + 2s + 4)}$$

$$= 1 + \frac{1}{3(s - 2)} - \frac{10}{3\left([s + 1]^2 + \left(\sqrt{3}\right)^2\right)} - \frac{4s}{3\left([s + 1]^2 + \left(\sqrt{3}\right)^2\right)}$$

since, by completing the squares:

$$s^2 + 2s + 4 = s^2 + 2s + 1 + 3 = [s + 1]^2 + \left(\sqrt{3}\right)^2$$

Questions