Binomial series

1 A square tray is tightly packed with a square array of ball bearings with no space left for movement. Further ball bearings are stacked up to form a pyramid. If the square tray holds n to a side how many ball bearings are there in the pyramid? If the tray were not square but an equilateral triangle capable of taking n ball bearings to a side how many ball bearings would there be in this pyramid?

You are given that $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ and $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$

Solutions Working

2 An international committee of 6 is to be formed from 7 Frenchmen and 4 Germans. In how many ways can this be done when the committee consists of:

(a) Exactly 2 Germans
(b) At least two Germans

Solutions Working

3 A rectangular metal plate has 4 holes drilled down each of the longer sides. Each hole is to be plugged with a coloured grommet. Of the eight grommets available two are blue, one pale and the other dark, and must go on the left-hand side and one is red and must go on the right-hand side, the remaining five grommets are each of a different shade of yellow.

In how many different ways can the grommets be arranged in the plate?

Solutions Working

4 A lock on a briefcase consists of three rings each marked with 12 different letters. Only one combination of three letters will permit the lock to be opened. How many ways it is possible to make an unsuccessful attempt at opening the lock?

Solutions Working

Solutions

1 $\dfrac{n(n+1)(2n+1)}{6}$ and $\dfrac{n(n+1)(n+2)}{6}$

Questions Working

2 (a) 210
(b) 371

Questions Working

3 5760

Questions Working

4 1727

Questions Working

Working

1 There are n^2 ball bearings in the bottom layer of the pyramid. The next layer is also a square with $n-1$ ball bearings to a side so that there are $(n-1)^2$ ball bearings in the second layer. This proceeds up to the uppermost layer of the pyramid where there is just one ball bearing. Therefore, the total number of ball bearings in the pyramid is:

$$n^2+(n-1)^2+(n-2)^2+\ldots+2^2+1^2=\sum_{r=1}^{n}r^2=\frac{n(n-1)(2n+1)}{6}$$

If the tray is an equilateral triangle then the number of ball bearings in the bottom layer is:

$$n+(n-1)+(n-2)+\ldots+2+1=\sum_{r=1}^{n}r=\frac{n(n-1)}{2}=\frac{1}{2}(n^2+n)$$

In the second layer the number of ball bearings is:

$$(n-1)+(n-2)+\ldots+2+1=\sum_{r=1}^{n-1}r=\frac{(n-1)n}{2}=\frac{1}{2}\left([n-1]^2+[n-1]\right)$$

The next layer contains:

$$(n-2)+(n-3)+\ldots+2+1=\sum_{r=1}^{n-2}r=\frac{(n-2)(n-1)}{2}$$
$$=\frac{1}{2}\left([n-2]^2+[n-2]\right)$$

Therefore the number of ball bearings in the entire pyramid is:

$$\sum_{r=1}^{n} r + \sum_{r=1}^{n-1} r + \sum_{r=1}^{n-2} r + \ldots + \sum_{r=1}^{2} r + \sum_{r=1}^{1} r = \frac{1}{2}\left(\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r\right)$$

where

$$\frac{1}{2}\left(\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r\right) = \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{1}{2}\left(\frac{n(n+1)}{2}\right)$$
$$= \frac{n(n+1)(2n+4)}{12}$$
$$= \frac{n(n+1)(n+2)}{6}$$

Questions Solutions

---

2 (a) If there are exactly 2 Germans on the committee then there are 4 Frenchmen on the committee. The 2 Germans can be selected from the 4 in:

$${}^{4}C_2 = \frac{4!}{(4-2)!2!} = \frac{24}{2 \times 2} = 6 \text{ different ways}$$

The 4 Frenchmen can be selected in:

$${}^{7}C_4 = \frac{7!}{(7-4)!4!} = \frac{7!}{3!4!} = \frac{5040}{6 \times 24} = 35 \text{ different ways}$$

Therefore the committee can be constituted in $6 \times 35 = 210$ different ways.

(b) If the committee is to have at least 2 Germans then it can have 2, 3 or 4 Germans.

Exactly 2 Germans gives ${}^{4}C_2 \times {}^{7}C_4 = 210$ different ways

Exactly 3 Germans gives ${}^{4}C_3 \times {}^{7}C_3 = \frac{4!}{1!3!} \times \frac{7!}{4!3!}$
$= 140$ different ways

Exactly 4 Germans gives ${}^{4}C_4 \times {}^{7}C_2 = \frac{4!}{0!4!} \times \frac{7!}{5!2!}$
$= 21$ different ways

Therefore the committee can be constituted in $210 + 140 + 21 = 371$ different ways.

Questions Solutions

---

3 The first of the two blue grommets that can only be put on the left-hand side can be put in one of 4 holes and then the second can then be put in one of 3 remaining holes – that is 4×3 arrangements. The red grommet can be placed in one of four holes so that the two blue grommets and the single red grommet can be arranged in $4 \times 3 \times 4$ different ways. This leaves 5 holes to be plugged with the yellow grommets. These can be put in 5! different arrangements for each of the arrangements of the blue and red grommets. Therefore there are $4 \times 3 \times 4 \times 5! = 5760$ different arrangements.

Questions Solutions

4 The first ring can be rotated to reveal a letter in 12 different ways. For each of these ways there are 12 ways of selecting the second letter so there are $12 \times 12 = 144$ ways of selecting the first two letters. For each of these selections there are 12 ways of selecting the third letter so there is a total of:

$12 \times 12 \times 12 = 1728$ different combinations of three letters

Only one will open the briefcase so there are 1727 that will not.

Questions Solutions