

# Functions

- 1** One component of a machine vibrates under normal operation such that:

$$f(t) = \cos(t)$$

where  $t$  measures the time and  $f(t)$  measures the position of the centre of gravity of the component. A number of loads can each be added to the machine, each one of which superimposes another vibration of the forms:

- (a)  $g(t) = \sin t$
- (b)  $g(t) = \sin 2t$
- (c)  $g(t) = \sin \sqrt{2}t$

Plot the graphs of  $f(t) + g(t)$  against  $t$  for  $-10 \leq t \leq 10$  with a step interval of 1 using your spreadsheet and describe the resultant behaviour or the vibration in comparison to the original, unloaded behaviour:

Are the functions  $f(t) + g(t)$  odd or even?  
Do they display a repeated pattern?

Working

- 2** If a car spring is compressed and then released it will vibrate up and down freely unless it is constrained by a damper.

- (a) Describe how the amplitude of this motion can be modelled by the product of two functions  $f(t) = e^{-kt}$  and  $g(t) = \cos t$  where  $t$  is a time parameter.
- (b) By drawing a suitable graph show the damping effect for  $k = 0.1$ ,  $k = 1$  and  $k = 10$ .

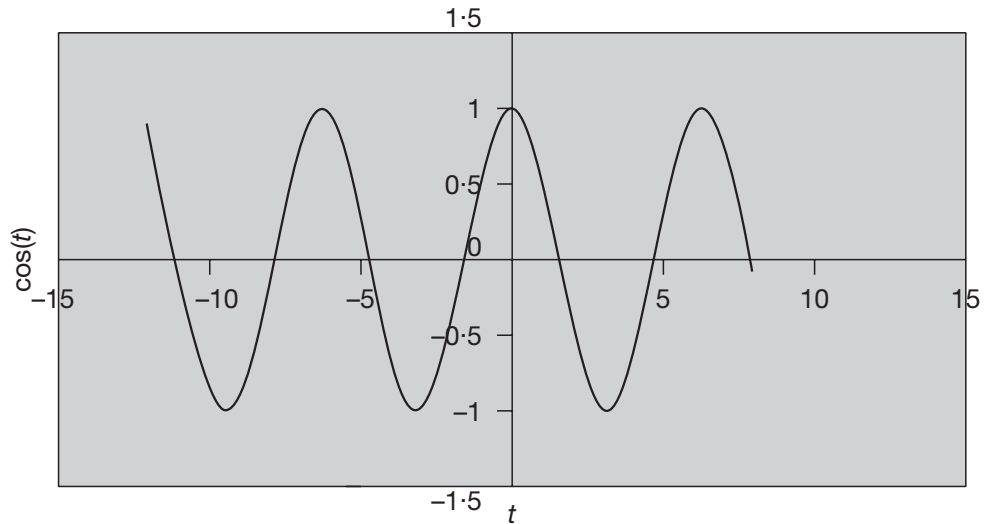
Working

- 3** Construct the compositions of  $f(t) = e^{-t}$  and  $g(t) = \cos t$  for non-negative  $t$ . Describe the behaviour of their outputs.

Working

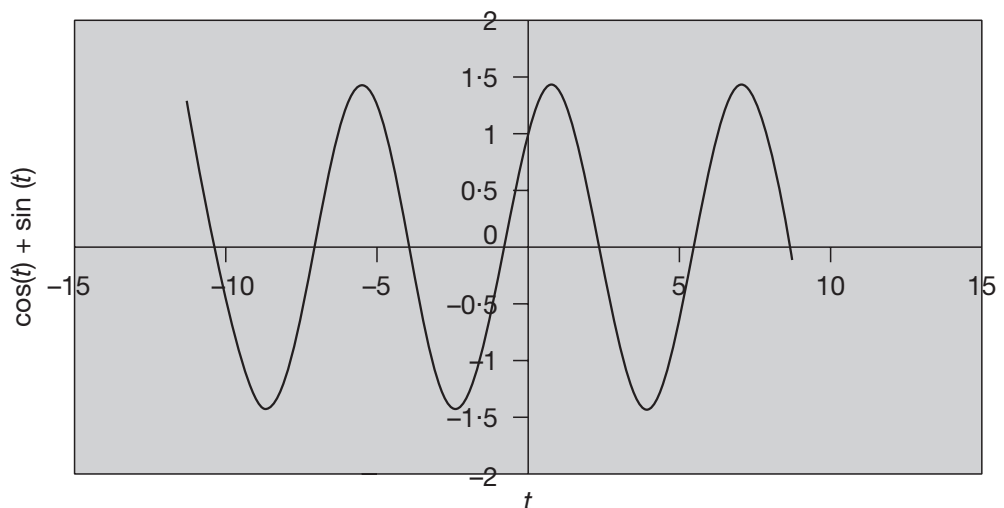
## Working

- 1 The original behaviour is as shown below.



The graph is symmetric about the vertical axis demonstrating that  $f(-t) = \cos(-t) = \cos t = f(t)$ , so that  $f(t)$  is an even function. There is a repeated pattern which repeats every  $2\pi$  radians. This is the period of the function.

- (a) The graph of  $f(t) + g(t) = \cos t + \sin t$  is shown below. This is identical in overall shape to the graph of  $f(t) = \cos t$  but with a greater amplitude and shifted to the right.



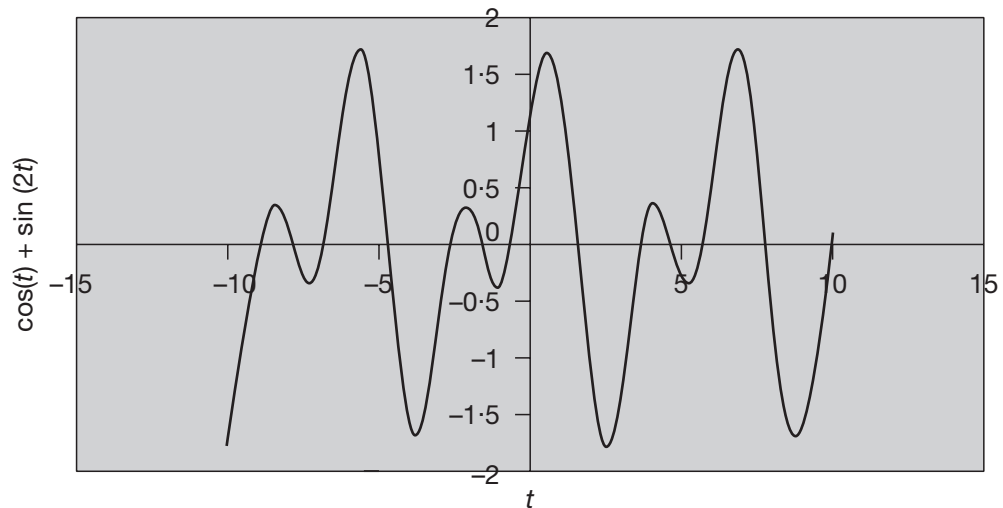
Since

$$\begin{aligned}\cos t + \sin t &= \sqrt{2} \left( \frac{1}{\sqrt{2}} \times \cos t + \frac{1}{\sqrt{2}} \times \sin t \right) \\ &= \sqrt{2} (\cos(\pi/4) \cos t + \sin(\pi/4) \sin t) \\ &= \sqrt{2} \cos(t - \pi/4)\end{aligned}$$

we see that the amplitude is  $\sqrt{2}$  times the value of the amplitude of the original function and the graph is shifted to the right by  $\pi/4$  radians.

Again the graph displays a repeated pattern but because of the shift it is no longer symmetrical about the vertical axis and  $f(t) + g(t)$  is, therefore, neither odd nor even. The period remains the same at  $2\pi$ .

- (b) The graph of  $f(t) + g(t) = \cos t + \sin 2t$  is shown below. This is no longer identical in overall shape to the graph of  $f(t) = \cos t$ .

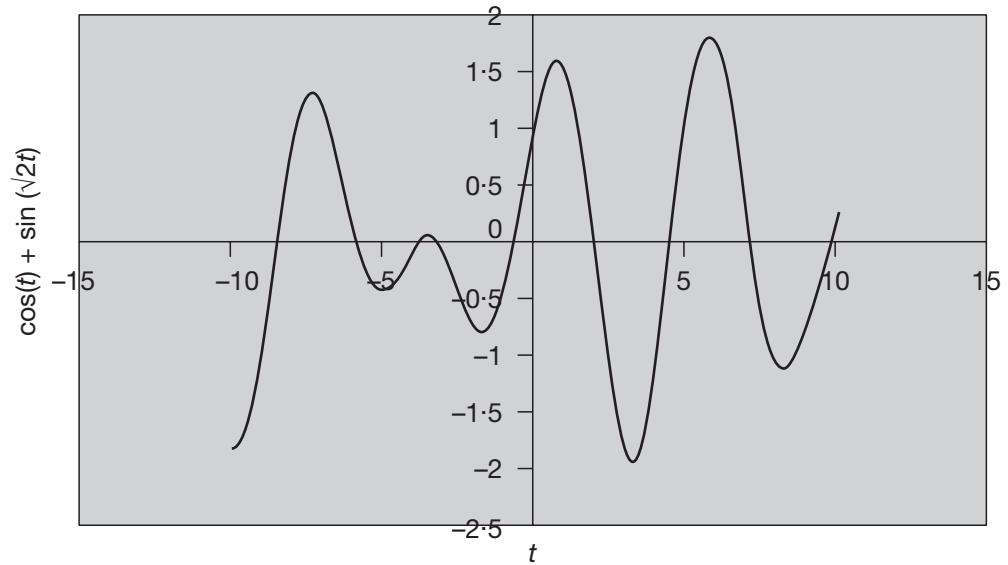


Here the shape is more involved than the simple sinusoidal shape but it still has echoes of the original shape and it still displays a repeated pattern. Since:

$$\cos t + \sin 2t = \cos t + 2 \sin t \cos t = \cos t(1 + 2 \sin t)$$

it is evident that the period remains  $2\pi$ . The graph also demonstrates that  $f(t) + g(t)$  is neither odd nor even.

- (c) The graph of  $f(t) + g(t) = \cos t + \sin \sqrt{2}t$  is shown below. This is no longer even similar in overall shape to the graph of  $f(t) = \cos t$ . Indeed, it no longer displays any repeated pattern.



This emphasises an important fact. The graph of  $h(t) = \cos at + \sin bt$  will only display a repeated pattern if the ratio  $a/b$  is a rational number<sup>4</sup>; in our example  $\sqrt{2}$  is irrational.

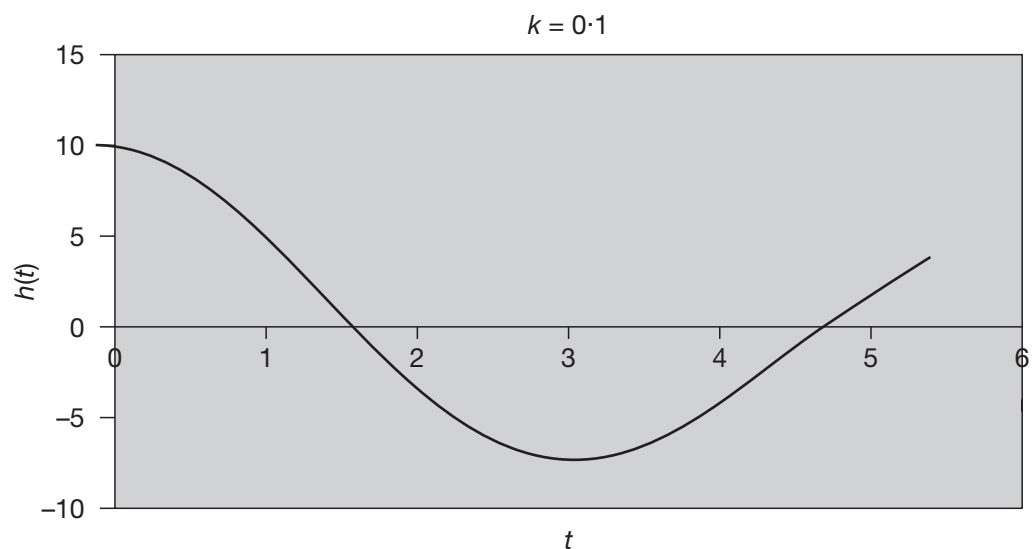
Questions

**2**

- (a) The free, periodic motion can be modelled by the periodic function  $A \cos t$  where  $A$  is the amplitude of vibration. Since  $f(t) = e^{-kt}$  decays from a value of 1 at  $t = 0$  and approaches zero as  $t$  increases then:

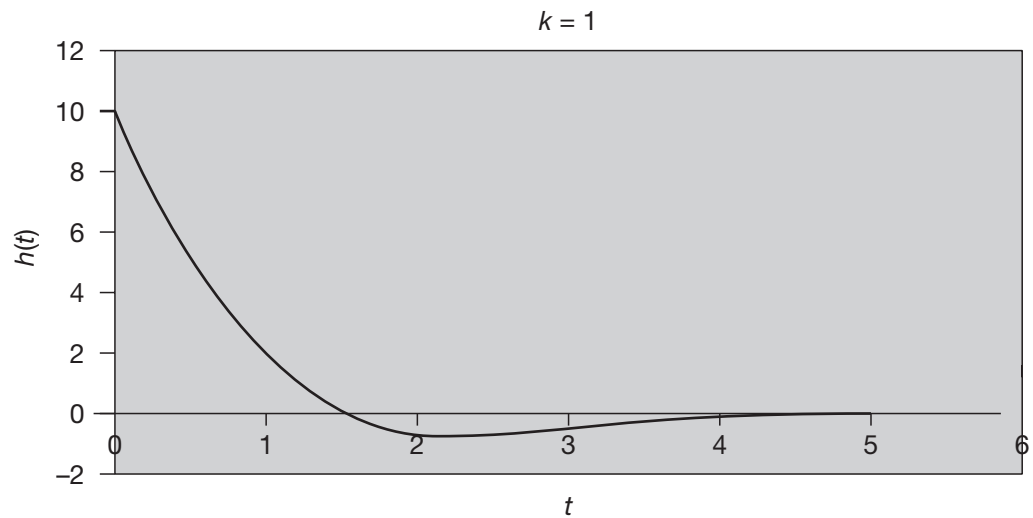
$h(t) = Ae^{kt} \cos t$  represents a damped periodic motion whose initial amplitude at  $t = 0$  is  $A$ .

- (b) The graph for  $k = 0.1$  is given as:



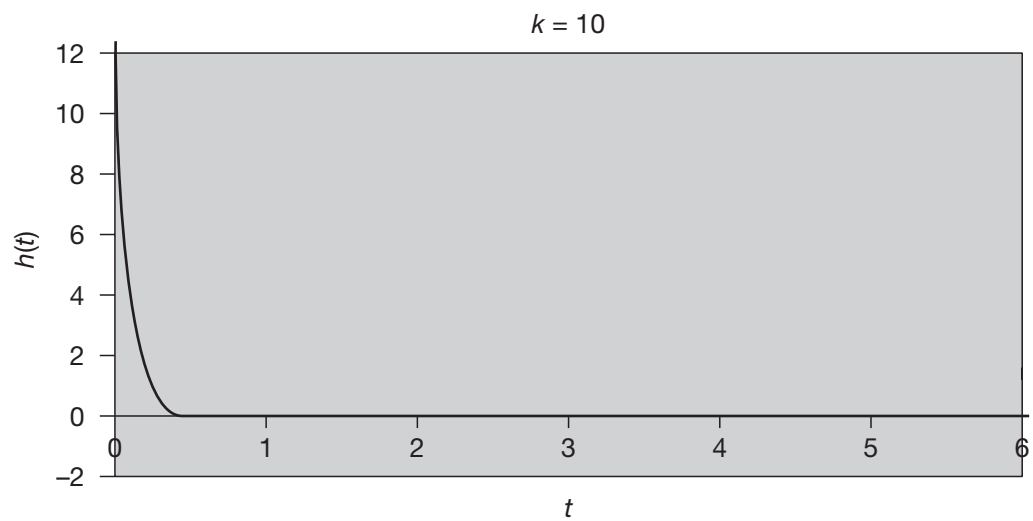
This demonstrates a decreasing amplitude of vibration – damped vibration.

The graph for  $k = 1$  is given as:



This demonstrates damping that reduces the vibration to a single bounce and then a rapid decay to zero amplitude.

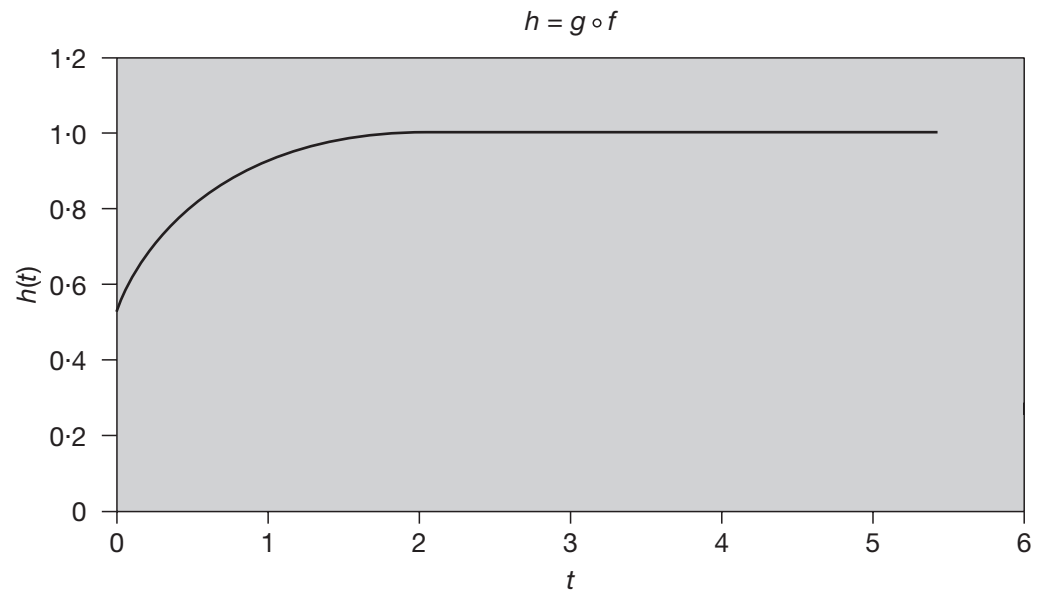
The graph for  $k = 10$  is given as:



This demonstrates damping that completely inhibits vibration producing a rapid decay to zero amplitude.

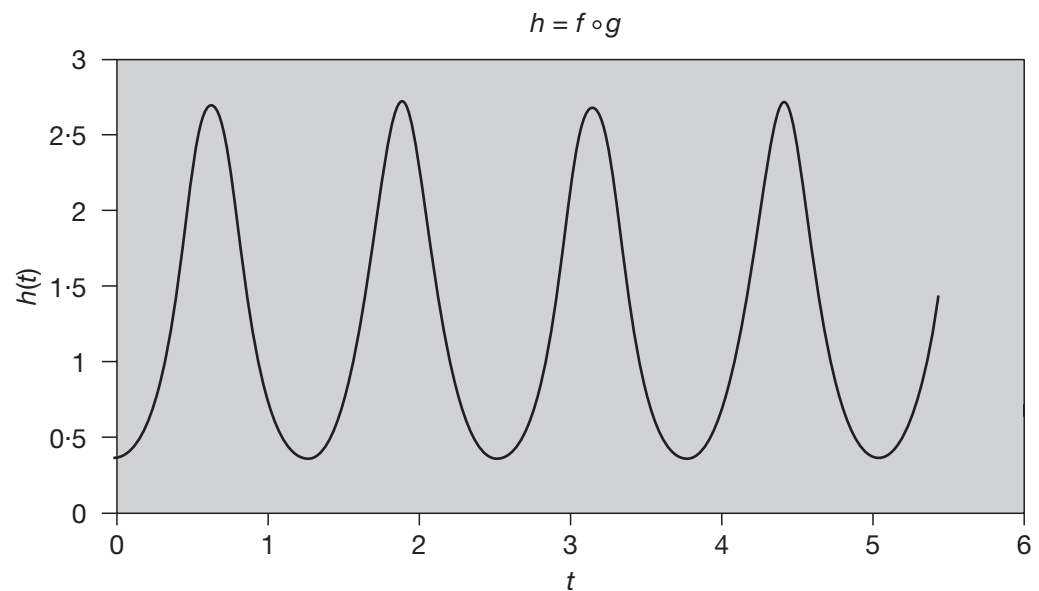
Questions

**3**  $h = g \circ f$  so that  $h(t) = g[f(t)] = \cos(e^{-t})$



Here, as  $t$  increases so  $e^{-t}$  approaches zero and so  $\cos t$  approaches  $\cos 0 = 1$ .

$h = f \circ g$  so that  $h(t) = f[g(t)] = e^{-\cos t}$



Here, as  $t$  increases so  $\cos t$  ranges between  $\pm 1$  and so  $h(t)$  ranges between  $e$  and  $e^{-1}$  producing the periodic behaviour shown.

Questions