Differentiation

1 The pressure and volume of an ideal gas undergoing an adiabatic change obey the rule:

$$PV^{\gamma} = k$$

where γ and k are constants for the given gas. If, during an adiabatic change, the pressures are plotted against the corresponding volumes show that the slope of the graph is given as:

$$\frac{\mathrm{d}P}{\mathrm{d}V} = -\gamma\frac{P}{V}$$

Working

2 Kirchoff's second law states that the sum of the voltage drops in a closed electrical circuit is equal to the applied emf. If resistance R ohms, inductance L henrys and capacitance C farads are connected in series to an applied emf $E(t)$ volts show that the circuit can be described by the differential equation:

$$Li''(t) + Ri'(t) + \frac{i(t)}{C} = E'(t)$$

where $i(t)$ is the current (amperes) flowing in the circuit and $q(t)$ is the charge (coulombs) on the capacitor at time t seconds and where $i(t) = q'(t)$.

Working

3 The spectrum of black body radiation is given as:

$$u(\lambda) = \frac{8\pi c}{\lambda^5}\frac{1}{e^{hc/\lambda kT} - 1}$$

where λ is the wavelength of the radiation, $u(\lambda)$ is the energy density between wavelednngths λ and $\lambda + d\lambda$, h, c, T are Plank's constant, the speed of light and the absolute temperature respectively and k is the Boltzman constant. Show that $u'(\lambda) = 0$ when $\frac{x}{1 - e^{-x}} - 5 = 0$ where $x = hc/\lambda kt$. Show further that by plotting the graph of $f(x) = \frac{x}{1 - e^{-x}} - 5$ on a spreadsheet for $-10 \leq x \leq 20$ that $\lambda = \frac{hc}{5kT}$ is a good first approximation to λ when $u'(\lambda) = 0$.

Working

Working

1 $PV^{\gamma} = k$ and so $P = kV^{-\gamma}$

Consequently:

$$\begin{aligned}\frac{dP}{dV} &= \frac{dkV^{-\gamma}}{dV}\\ &= -k\gamma V^{-\gamma-1}\\ &= -\gamma(kV^{-\gamma})V^{-1}\\ &= -\gamma\frac{P}{V}\end{aligned}$$

Questions

2 When a changing current traverses an inductance it induces a back-emf that causes a potential drop over the inductance. By Faraday's law this is:

$$Li'(t)$$

When a current traverses a resistance it generates heat, thereby losing energy from the circuit. This evidences itself as a potential drop across the resistance. By Ohm's law this is:

$$Ri(t)$$

When a source of current is connected to a capacitance, the capacitance stores energy from the circuit. This evidences itself as a potential drop over the capacitance given as:

$$\frac{q(t)}{C}$$

Kirchoff's law states that the emf applied to a circuit is equal to the potential drops over the various components in the circuit. For an LCR-series circuit this means that if $E(t)$ is the applied emf:

$$Li'(t) + Ri(t) + \frac{q(t)}{C} = E(t)$$

If this equation is differentiated throughout we find that:

$$Li''(t) + Ri'(t) + \frac{q'(t)}{C} = E'(t)$$

Now, the current is the rate of change of charge so that $q'(t) = i(t)$. This gives the equation:

$$Li''(t) + Ri'(t) + \frac{i(t)}{C} = E'(t)$$

Questions

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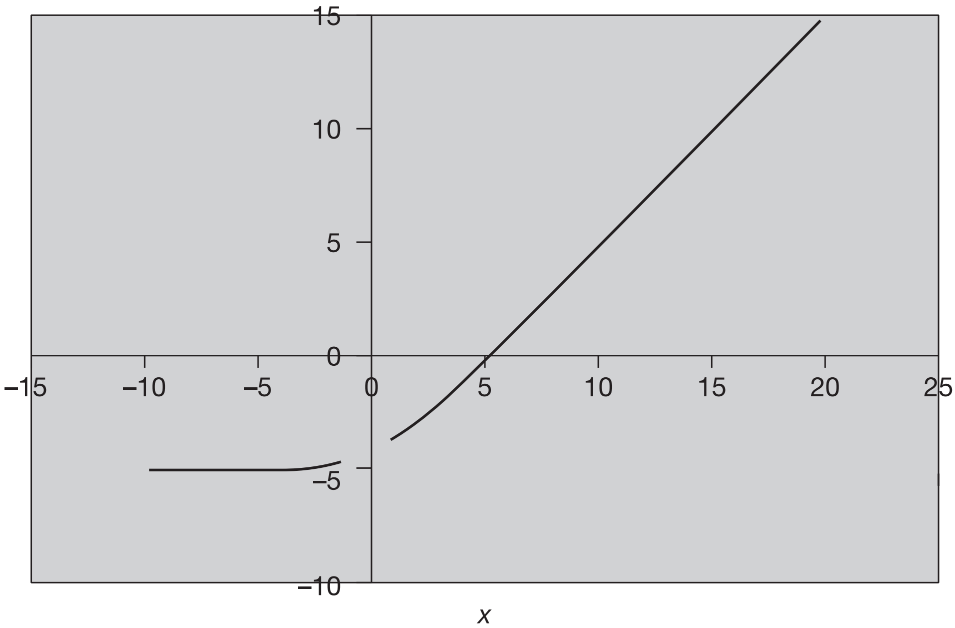
3 $u(\lambda) = \dfrac{8\pi c}{\lambda^5}\dfrac{1}{e^{hc/\lambda kT} - 1}$ therefore:

$$\begin{aligned}
u'(\lambda) &= -5\frac{8\pi c}{\lambda^6}\frac{1}{e^{hc/\lambda kT} - 1} - \frac{8\pi c}{\lambda^5}\frac{(-hc/\lambda^2 kT)e^{hc/\lambda kT}}{(e^{hc/\lambda kT} - 1)^2} \\
&= -5\frac{8\pi c}{\lambda^6}\frac{1}{e^{hc/\lambda kT} - 1} + \frac{8\pi c}{\lambda^6}\frac{(hc/\lambda kT)e^{hc/\lambda kT}}{(e^{hc/\lambda kT} - 1)^2} \\
&= -5\frac{8\pi c}{\lambda^6}\frac{1}{e^x - 1} + \frac{8\pi c}{\lambda^6}\frac{(x)e^x}{(e^x - 1)^2} \quad \text{where } x = hc/\lambda kT \\
&= \frac{8\pi c}{\lambda^6}\frac{1}{e^x - 1}\left(\frac{xe^x}{e^x - 1} - 5\right) \\
&= \frac{8\pi c}{\lambda^6}\frac{1}{e^x - 1}\left(\frac{x}{1 - e^{-x}} - 5\right) \\
&= 0
\end{aligned}$$

when

$$\frac{x}{1 - e^{-x}} - 5 = 0$$

A plot of $f(x) = \dfrac{x}{1 - e^{-x}} - 5$ will show that $x = 5$ is a good approximate solution to the equation $f(x) = 0$ [notice that $x = 0$ is not in the domain of the function].



Questions