Integration

1 An insulating rod of length 2 m and thin enough to be approximated by a line lies in an environment that induces a charge on its surface. If the rod lies along the z-axis between $z = 2$ m and $z = 4$ m and is subject to a charge density:

$\rho = 5z^3$ (μC/m) between the values $z = 2$ and $z = 4$ and

$\rho = 0$ (μC/m) outside that range,

find the total charge.

Solutions Working

2 The moment of inertia about its axis of symmetry of a rotating conical plug of vertical height h can be found by evaluating the integral:

$$I_x = \frac{\pi\rho}{2}\int_0^h [f(x)]^4 \mathrm{d}x$$

where the x-axis is taken to be the axis of symmetry with the vertex at $x = 0$ where ρ is the density of the material comprising the plug and where $y = f(x)$ is the projection of the profile of the cone in the xy plane. Find the moment of inertia if $h = 15$ cm, $\rho = 0.05$ kg cm^{-3} and where:

(a) $f(x) = 3x/5$
(b) $f(x) = x^2/60\pi$

Solutions Working

3 The moment of inertia of an object about an axis of rotation is given as:

$I = Mk^2$

where M is the mass of the object and k is the radius of gyration about the axis of rotation. A thin spindle of variable density and length L which at rest has one end at $x = 0$ and the other end at $x = L$ is set in rotary motion about an axis vertical to the spindle and passing through $x = 0$. The moment of inertia of the spindle in this configuration can found from the integral:

$I = \int_0^L \rho(x)x^2\,dx$ where $\rho(x)$ is the density and $M = \int_0^L \rho(x)\,dx$

Find the radius of gyration of the spindle if:

(a) $L = 32$ cm and $\rho(x) = 0{\cdot}1$ kg cm^{-3}
(b) $L = 17$ cm and $\rho(x) = (0{\cdot}1)(20 - x)$ kg cm^{-3}

Solutions Working

4 The growth rate of a certain culture is given by the equation:

$$\frac{dm}{dt} = 6 + 3\sin 2\pi t$$

where m is the mass in grams after t days. If the mass at time $t = 0$ is 30 grams, find the mass after 8 days.

Solutions Working

Solutions

1 300 μC

Questions Working

2 (a) $4{\cdot}921\pi$ kg cm^2 to 1 dp
(b) $0{\cdot}266$ kg cm^2 to 3dp

Questions Working

3 (a) $k = 18{\cdot}48$ cm to 2 dp
(b) $k = 7{\cdot}79$ cm to 2 dp

Questions Working

4 78 g

Questions Working

Working

1

$$\text{Total charge} = \int_{-\infty}^{\infty} \rho(z)\mathrm{d}z = \int_2^4 5z^3\mathrm{d}z$$

$$= \left[\frac{5z^4}{4}\right]_2^4 = (5 \times 64) - (5 \times 4) = 300\ \mu\text{C}$$

Questions Solutions

2

(a) If $f(x) = 3x/5$ then:

$$I_x = \frac{\pi\rho}{2}\int_0^h [f(x)]^4\,\mathrm{d}x$$

$$= \frac{\pi 0{\cdot}05}{2}\int_0^{15} [3x/5]^4\,\mathrm{d}x$$

$$= \frac{81\pi 0{\cdot}05}{2 \times 625}\int_0^{15} x^4\,\mathrm{d}x$$

$$= 0{\cdot}00324\pi\left[\frac{x^5}{5}\right]_0^{15}$$

$$= 0{\cdot}00324\pi\left(\frac{15^5}{5}\right) = 492{\cdot}1\pi\ \text{kg cm}^2\ \text{to 1 dp}$$

(b) If $f(x) = x^2/60\pi$ then:

$$I_x = \frac{\pi\rho}{2}\int_0^h \left[x^2/60\pi\right]^4\,\mathrm{d}x$$

$$= \frac{0{\cdot}05}{2 \times \pi^3 \times 60^4}\int_0^{15} x^8\,\mathrm{d}x$$

$$= \frac{0{\cdot}05}{2 \times \pi^3 \times 60^4}\left[\frac{x^9}{9}\right]_0^{15}$$

$$= \frac{0{\cdot}05}{3\pi^3 \times 60^4}\left(\frac{15^9}{9}\right) = 02{\cdot}266\ \text{kg cm}^2\ \text{to 3 dp}$$

Questions Solutions

3 (a) $L = 32$ cm and $\rho = 0{\cdot}1$ kg cm^{-3}

$$
\begin{aligned}
I &= \int_0^L \rho(x)x^2\,dx \\
&= 0{\cdot}1\int_0^{32} x^2\,dx \\
&= 0{\cdot}1\left[\frac{x^3}{3}\right]_0^{32} \\
&= 0{\cdot}1\left(\frac{32^3}{3}\right) = 1092{\cdot}267 \text{ to 3 dp}
\end{aligned}
$$

$$
\begin{aligned}
M &= \int_0^L \rho(x)\,dx \\
&= 0{\cdot}1\int_0^{32} dx \\
&= 0{\cdot}1\,[x]_0^{32} = 0{\cdot}1(32) = 3{\cdot}2
\end{aligned}
$$

Therefore, since $I = Mk^2$ and $I = 1092{\cdot}267 = 3{\cdot}2k^2$ we see that $k^2 = 1092{\cdot}267/3{\cdot}2 = 341{\cdot}333$. So $k = 18{\cdot}48$ cm to 2 dp.

(b) $L = 17$ cm and $\rho(x) = (0{\cdot}1)(20 - x)$ kg cm^{-3}

$$
\begin{aligned}
I &= \int_0^L \rho(x)x^2\,dx \\
&= 0{\cdot}1\int_0^{17} (20 - x)x^2\,dx \\
&= 0{\cdot}1\left[20\frac{x^3}{3} - \frac{x^4}{4}\right]_0^{17} \\
&= 0{\cdot}1\left(20\frac{17^3}{3} - \frac{17^4}{4}\right) = 1187{\cdot}308 \text{ to 3 dp}
\end{aligned}
$$

$$
\begin{aligned}
M &= \int_0^L \rho(x)\,dx \\
&= 0{\cdot}1\int_0^{17} (20 - x)\,dx \\
&= 0{\cdot}1\left[20x - \frac{x^2}{2}\right]_0^{17} \\
&= 0{\cdot}1\left(20 \times 17 - \frac{17^2}{2}\right) = 19{\cdot}55
\end{aligned}
$$

Therefore, since $I = Mk^2$ and $I = 1187{\cdot}308 = 19{\cdot}55k^2$ we see that $k^2 = 1187{\cdot}308/19{\cdot}55 = 60{\cdot}731$. So $k = 7{\cdot}79$ cm to 2 dp.

Questions Solutions

4 Given $\frac{dm}{dt} = 6 + 3\sin 2\pi t$ and the fact that at time $t = 0$, $m = 30$ then integrating the equation yields:

$$\int \frac{dm}{dt}\,dt = \int dm = m = \int (6 + 3\sin 2\pi t)\,dt = 6t - 3\frac{\cos 2\pi t}{2\pi} + C$$

where C is the integration constant. Now at time $t = 0$, $m = 30$ so substitution into:

$$m = 6t - 3\frac{\cos 2\pi t}{2\pi} + C$$

yields:

$$30 = -\frac{3}{2\pi} + C$$

Therefore $C = 30 + \frac{3}{2\pi}$ and the final equation for the mass is

$$m = 30 + 6t - \frac{3}{2\pi}(\cos 2\pi t - 1)$$

After 8 days

$$m = 30 + 6 \times 8 - \frac{3}{2\pi}(\cos 2 \times 8\pi - 1) = 78 - \frac{3}{2\pi}(1 - 1)$$
$$= 78 \text{ grams}$$

Questions Solutions