# Complex numbers 2

**1** Given that the electric field $\mathbf{E} = \mathrm{Re}\left[\hat{\mathbf{E}}e^{j\omega t}\right]$ and the magnetic field $\mathbf{H} = \mathrm{Re}\left[\hat{\mathbf{H}}e^{j\omega t}\right]$ of a wave travelling in free space are given as:

$$\mathbf{E} = 25\cos\left(\omega t + \frac{2\pi}{5}z\right)\mathbf{i}\ \text{(V/m) and } \mathbf{H} = \frac{25}{2\pi}\cos\left(\omega t + \frac{2\pi}{5}z\right)\mathbf{j}\ \text{(A/m)}$$

Find the magnitude and direction of the phasor Poynting vector $\hat{\mathbf{S}} = \hat{\mathbf{E}} \times \hat{\mathbf{H}}^*$ where $\hat{\mathbf{E}}$ and $\hat{\mathbf{H}}$ are the electric and magnetic phasors respectively.

[*Hint*: A phasor is the complex number that embodies the magnitude and phase of a sinusoidal expression]

Solution Working

**2** For electric and magnetic fields that vary sinusoidally in time show that the time-averaged Poynting vector $\langle \mathbf{S} \rangle$ is given as:

$$\langle \mathbf{S} \rangle = \frac{1}{2}\mathrm{Re}\left[\hat{\mathbf{S}}\right]$$

where $\hat{\mathbf{S}}$ is the phasor Poynting vector that is defined in terms of the electric and magnetic field phasors as:

$$\hat{\mathbf{S}} = \hat{\mathbf{E}} \times \hat{\mathbf{H}}^*$$

given that $\mathbf{E} = \mathrm{Re}\left(\hat{\mathbf{E}}e^{j\omega t}\right)$ where the vector with constant complex magnitude $\hat{\mathbf{E}}$ is defined as the electric field phasor and a similar definition applies for the magnetic field phasor.

*Hint*: The time average of a sinusoidal signal $Ae^{j\omega t}$ is given as:

$$\langle Ae^{j\omega t} \rangle = \frac{\int_0^T Ae^{j\omega t}\,\mathrm{d}t}{\int_0^T \mathrm{d}t} \quad \text{where } T = \frac{2\pi}{\omega}$$

Working

## Solution

1 $\hat{\mathbf{S}} = \dfrac{625}{2\pi}\mathbf{k}$ W/m2

Questions Working

## Working

1 Since $\mathbf{E} = \mathrm{Re}\left[\hat{\mathbf{E}}e^{j\omega t}\right] = 25\cos\left(\omega t + \dfrac{2\pi}{5}z\right)\mathbf{i} = \mathrm{Re}\left[25e^{2\pi jz/5}\,\mathbf{i}\,e^{j\omega t}\right]$

we see that $\hat{\mathbf{E}} = 25e^{2\pi jz/5}\,\mathbf{i}$

and similarly $\mathbf{H} = \mathrm{Re}\left[\hat{\mathbf{H}}e^{j\omega t}\right] = \dfrac{25}{2\pi}\cos\left(\omega t + \dfrac{2\pi}{5}z\right)\mathbf{j} = \mathrm{Re}\left[\dfrac{25}{2\pi}e^{2\pi jz/5}\,\mathbf{j}\,e^{j\omega t}\right]$

so that $\hat{\mathbf{H}} = \dfrac{25}{2\pi}e^{2\pi jz/5}\mathbf{j}$. These give rise to the phasor Poynting vector:

$$\begin{aligned}\hat{\mathbf{S}} &= \hat{\mathbf{E}} \times \hat{\mathbf{H}}^* \\ &= \left(25e^{2\pi jz/5}\,\mathbf{i}\right) \times \left(\frac{25}{2\pi}e^{2\pi jz/5}\,\mathbf{j}\right)^* \\ &= \frac{25 \times 25}{2\pi}e^{2\pi jz/5}e^{-2\pi jz/5}\,\mathbf{i} \times \mathbf{j} \\ &= \frac{625}{2\pi}\mathbf{k}\ \mathrm{W/m^2}\end{aligned}$$

Questions Solution

**2** The electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$ are given in terms of the phasors $\hat{\mathbf{E}}$ and $\hat{\mathbf{H}}$ and their complex conjugates $\hat{\mathbf{E}}^*$ and $\hat{\mathbf{H}}^*$ as:

$$\mathbf{E} = \mathrm{Re}\left[\hat{\mathbf{E}}e^{j\omega t}\right] = \frac{1}{2}\left(\hat{\mathbf{E}}e^{j\omega t} + \hat{\mathbf{E}}^* e^{-j\omega t}\right)$$

and

$$\mathbf{H} = \mathrm{Re}\left[\hat{\mathbf{H}}e^{j\omega t}\right] = \frac{1}{2}\left(\hat{\mathbf{H}}e^{j\omega t} + \hat{\mathbf{H}}^* e^{-j\omega t}\right)$$

The Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ and the Poynting phasor $\hat{\mathbf{S}} = \hat{\mathbf{E}} \times \hat{\mathbf{H}}^*$ are given as:

$$\begin{aligned}\mathbf{S} &= \mathbf{E} \times \mathbf{H}\\ &= \frac{1}{2}\left(\hat{\mathbf{E}}e^{j\omega t} + \hat{\mathbf{E}}^* e^{-j\omega t}\right) \times \frac{1}{2}\left(\hat{\mathbf{H}}e^{j\omega t} + \hat{\mathbf{H}}^* e^{-j\omega t}\right)\\ &= \frac{1}{4}\left(\hat{\mathbf{E}} \times \hat{\mathbf{H}}^* + \hat{\mathbf{E}}^* \times \hat{\mathbf{H}}\right) + \frac{1}{4}\left(\hat{\mathbf{E}} \times \hat{\mathbf{H}}e^{j2\omega t} + \hat{\mathbf{E}}^* \times \hat{\mathbf{H}}^* e^{-j2\omega t}\right)\\ &= \frac{1}{4}\left(\hat{\mathbf{E}} \times \hat{\mathbf{H}}^* + \left[\hat{\mathbf{E}} \times \hat{\mathbf{H}}^*\right]^*\right) + \frac{1}{4}\left(\hat{\mathbf{E}} \times \hat{\mathbf{H}}e^{j2\omega t} + \left[\hat{\mathbf{E}} \times \hat{\mathbf{H}}\right]^* e^{-j2\omega t}\right)\end{aligned}$$

because $\left(\hat{\mathbf{E}} \times \hat{\mathbf{H}}^*\right)^* = \hat{\mathbf{E}}^* \times \hat{\mathbf{H}}$ and $\hat{\mathbf{E}}^* \times \hat{\mathbf{H}}^* = \left(\hat{\mathbf{E}} \times \hat{\mathbf{H}}\right)^*$ therefore:

$$\mathbf{S} = \frac{1}{2}\mathrm{Re}\left(\hat{\mathbf{E}} \times \hat{\mathbf{H}}^*\right) + \frac{1}{2}\mathrm{Re}\left(\hat{\mathbf{E}} \times \hat{\mathbf{H}}e^{j2\omega t}\right)$$

Now time average:

$$\langle\mathbf{S}\rangle = \left\langle\frac{1}{2}\mathrm{Re}\left(\hat{\mathbf{E}} \times \hat{\mathbf{H}}^*\right)\right\rangle + \left\langle\frac{1}{2}\mathrm{Re}\left(\hat{\mathbf{E}} \times \hat{\mathbf{H}}e^{j2\omega t}\right)\right\rangle$$

Now:

$$\begin{aligned}\left\langle\frac{1}{2}\mathrm{Re}\left(\hat{\mathbf{E}} \times \hat{\mathbf{H}}^*\right)\right\rangle &= \frac{\displaystyle\int_0^T \frac{1}{2}\mathrm{Re}\left(\hat{\mathbf{E}} \times \hat{\mathbf{H}}^*\right)\mathrm{d}t}{\displaystyle\int_0^T \mathrm{d}t}\\ &= \frac{\displaystyle\frac{1}{2}\mathrm{Re}\left(\hat{\mathbf{E}} \times \hat{\mathbf{H}}^*\right)\int_0^T \mathrm{d}t}{\displaystyle\int_0^T \mathrm{d}t}\\ &= \frac{1}{2}\mathrm{Re}\left(\hat{\mathbf{E}} \times \hat{\mathbf{H}}^*\right)\\ &= \frac{1}{2}\mathrm{Re}\left(\hat{\mathbf{S}}\right)\end{aligned}$$

and:

$$\left\langle \frac{1}{2}\mathrm{Re}\left(\hat{\mathbf{E}} \times \hat{\mathbf{H}} e^{j2\omega t}\right) \right\rangle = \frac{\displaystyle\int_0^{2\pi/\omega} \frac{1}{2}\mathrm{Re}\left(\hat{\mathbf{E}} \times \hat{\mathbf{H}} e^{j2\omega t}\right) \mathrm{d}t}{\displaystyle\int_0^{2\pi/\omega} \mathrm{d}t}$$

$$= \frac{\frac{1}{2}\mathrm{Re}\left(\hat{\mathbf{E}} \times \hat{\mathbf{H}}\right)\left[\frac{e^{j2\omega t}}{j2\omega}\right]_0^{2\pi/\omega}}{2\pi/\omega}$$

$$= \frac{\frac{1}{2}\mathrm{Re}\left(\hat{\mathbf{E}} \times \hat{\mathbf{H}}\right)\left(e^{j4\pi} - 1\right)}{4\pi j}$$

$$= 0 \quad \text{since } e^{j4\pi} = 1$$

Therefore:

$$\langle \mathbf{S} \rangle = \frac{1}{2}\mathrm{Re}(\hat{\mathbf{S}}) + 0$$

Questions