Vectors

1 Coulomb's Law states that the force experienced in free space by a charge q located at $\mathbf{r} = (a, b, c)$ due to the presence of a charge q' located at $\mathbf{r}' = (a', b', c')$ is given by:

$$\mathbf{F}_{\mathrm{q}} = \frac{qq'}{4\pi\epsilon_0 R^2}\mathbf{n}_{\mathrm{R}} \text{ Newtons}$$

where $\mathbf{n}_{\mathrm{R}}$ is a unit vector directed along the straight line joining the two charges, R is the separation of the two charges in metres and ϵ_0 is the permittivity of free space. Find the force in terms of the coordinates of the two charges.

Solutions Working

2 Coulomb's Law states that the force experienced in free space by a charge q located at $\mathbf{r} = (a, b, c)$ due to the presence of a charge q' located at $\mathbf{r}' = (a', b', c')$ is given by:

$$\mathbf{F}_{\mathrm{q}} = \frac{qq'}{4\pi\epsilon_0 R^2}\mathbf{n}_{\mathrm{R}} \text{ Newtons}$$

where $\mathbf{n}_{\mathrm{R}}$ is a unit vector directed along the straight line joining the two charges, R is the separation of the two charges in metres and ϵ_0 is the permittivity of free space. Find the magnitude and the direction of the electrostatic force experienced by a charge $-200\ \mu$C situated at $(1, 2, 3)$ as a consequence of a charge $30\ \mu$C situated at $(0, 1, 1)$ if the permittivity of free space is given as $8{\cdot}854 \times 10^{-12}$ F/m.

Solutions Working

3 Four equal charges, of size 20 μC are placed at the corners of a square whose diagonal measures 12 m. Find the force on a charge of 100 μC placed at point P, 8 m above the centre of the square, given that the force experienced in free space by a charge q located at $\mathbf{r} = (a, b, c)$ due to the presence of a charge q' located at $\mathbf{r}' = (a', b', c')$ is given by:

$$\mathbf{F}_{\mathrm{q}} = \frac{qq'}{4\pi\epsilon_0 R^2}\mathbf{n}_{\mathrm{P}} \text{ Newtons}$$

where $\mathbf{n}_{\mathrm{P}}$ is a unit vector directed along the straight line joining the two charges, R is the separation of the two charges in metres and ϵ_0 is the permittivity of free space whose value is approximately $10^{-9}/36\pi$ F/m.

Solutions | Working

4 Let $\mathbf{S}$ be a vector that is sinusoidally time dependent and let $\hat{\mathbf{S}}$ be the phasor representation of $\mathbf{S}$. Show that the rms value of $\mathbf{S}$ is given by:

$$S_{\mathrm{rms}} = \sqrt{\frac{1}{2}\hat{\mathbf{S}}\cdot\hat{\mathbf{S}}^*}$$

Hint: The time average of a sinusoidal signal $Ae^{j\omega t}$ is given as:

$$\langle A\rangle = \frac{\int_0^T Ae^{j\omega t}\,\mathrm{d}t}{\int_0^T \mathrm{d}t} \quad \text{where} \quad T = \frac{2\pi}{\omega}$$

and

$$A_{\mathrm{rms}} = \sqrt{\langle A^2\rangle} \quad \text{is the root mean square of } A$$

Working

5 Force $\mathbf{F}_1 = 4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ acts through the point (5, 3, 1) and force $\mathbf{F}_2 = -6\mathbf{j} + 7\mathbf{k}$ acts through the point (−1, 0, 2).

(a) Find the total vector moment $\mathbf{M}$ of the two forces about the point (−2, 0, 1).
(b) Find the magnitude of $\mathbf{M}$.

Solutions | Working

Solutions

1

$$\mathbf{F}_q = \frac{qq'\left([a-a']\mathbf{i}+[b-b']\mathbf{j}+[c-c']\mathbf{k}\right)}{4\pi\epsilon_0\left([a-a']^2+[b-b']^2+[c-c']^2\right)^{3/2}}$$

Questions Working

2 Magnitude 8.988 Newtons (to 3 dp) in the direction

$$\mathbf{n}_R = \left(-1/\sqrt{6},\ -1/\sqrt{6},\ -2/\sqrt{6}\right)$$

Questions Working

3 $0{\cdot}9\mathbf{n}_\perp$ where $\mathbf{n}_\perp$ is a unit vector perpendicular to the face of the square.

Questions Working

5 (a) $9\,\mathbf{i}+3\mathbf{k}$
(b) $3\sqrt{10}$

Questions Working

Working

1

$$\mathbf{n}_R = \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$$

$$= \frac{\left([a-a']\mathbf{i}+[b-b']\mathbf{j}+[c-c']\mathbf{k}\right)}{\left([a-a']^2+[b-b']^2+[c-c']^2\right)^{1/2}}$$

and

$$R^2 = [a-a']^2+[b-b']^2+[c-c']^2$$

Therefore

$$\mathbf{F}_q = \frac{qq'}{4\pi\epsilon_0 r^2}\mathbf{n}_R$$

$$= \frac{qq'\left([a-a']\mathbf{i}+[b-b']\mathbf{j}+[c-c']\mathbf{k}\right)}{4\pi\epsilon_0\left([a-a']^2+[b-b']^2+[c-c']^2\right)^{3/2}}$$

Questions Solutions

2

$\mathbf{F}_q = \dfrac{qq'}{4\pi\epsilon_0 R^2}\mathbf{n}_R$ so that:

$$\left|\mathbf{F}_q\right| = \left|\dfrac{qq'}{4\pi\epsilon_0 R^2}\mathbf{n}_R\right| = \dfrac{|q||q'|}{4\pi\epsilon_0 R^2}|\mathbf{n}_R|$$

$$= \dfrac{|q||q'|}{4\pi\epsilon_0 R^2}$$

Now, $\mathbf{R} = \mathbf{r} - \mathbf{r}'$

$$= [1-0]\mathbf{i} + [2-1]\mathbf{j} + [3-1]\mathbf{k}$$

$$= \mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

and so $R^2 = [1]^2 + [1]^2 + [2]^2 = 6$

Therefore:

$$|\mathbf{F}_q| = \dfrac{|q||q'|}{4\pi\epsilon R^2}$$

$$= \dfrac{200\times 10^{-6}\times 30\times 10^{-6}}{4\pi\times 8{\cdot}854\times 10^{-12}\times 6}$$

$$= 8{\cdot}988 \text{ Newtons (to 3 dp)}$$

The direction of the attractive force is given as $-\mathbf{n}_R$ where:

$$-\mathbf{n}_R = -\dfrac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$$

$$= -\dfrac{(1,\ 1,\ 2)}{\sqrt{6}}$$

$$= \left(-1/\sqrt{6},\ -1/\sqrt{6},\ -2/\sqrt{6}\right)$$

Questions Solutions

3

A corner of the square, the centre of the square and point P form the vertices of a 3 : 4 : 5 right angled triangle. From the symmetry of the problem the resultant force will be perpendicular to the plane of the square. A unit vector $\mathbf{n}_\perp$ in such a direction is related to a unit vector joining a corner to point P, $\mathbf{n}_P$, by the equation:

$$\mathbf{n}_\perp = \left(\dfrac{4}{5}\right)\mathbf{n}_P$$

so that, since all four charges are identical and each 10 m distant from P:

$$\mathbf{F}_q = 4\times\dfrac{(100\times 10^{-6})(20\times 10^{-6})}{4\pi(10^{-9}/36\pi)10^2}\left(\dfrac{5}{4}\right)\mathbf{n}_\perp = 0{\cdot}9\,\mathbf{n}_\perp \text{ Newtons}$$

Questions Solutions

4 Let $\mathbf{S} = A\sin(\omega t + \alpha)\mathbf{i} + B\sin(\omega t + \beta)\mathbf{j} + C\sin(\omega t + \gamma)\mathbf{k}$ then, if $\langle\hat{\mathbf{S}}\rangle$ represents the time-average of $\hat{\mathbf{S}}$ over one period $T = 2\pi/\omega$, we see that:

$$S_{\text{rms}}{}^2 = \langle\mathbf{S}\cdot\mathbf{S}\rangle = A^2\langle\sin^2(\omega t + \alpha)\rangle + B^2\langle\sin^2(\omega t + \beta)\rangle + C^2\langle\sin^2(\omega t + \gamma)\rangle$$

Now:

$$\begin{aligned}
\langle\sin^2(\omega t + k)\rangle &= \frac{\displaystyle\int_0^{2\pi/\omega}\sin^2(\omega t + k)\,\mathrm{d}t}{\displaystyle\int_0^{2\pi/\omega}\mathrm{d}t} \\
&= \frac{\dfrac{1}{2}\displaystyle\int_0^{2\pi/\omega}[1 - \cos 2(\omega t + k)]\,\mathrm{d}t}{2\pi/\omega} \\
&= \frac{\dfrac{1}{2}\left[t - \dfrac{\sin 2(\omega t + k)}{2\omega}\right]_0^{2\pi/\omega}}{2\pi/\omega} \\
&= \frac{\dfrac{1}{2}\left(2\pi/\omega - \dfrac{\sin 2k}{2\omega} + \dfrac{\sin(4\pi + 2k)}{2\omega}\right)}{2\pi/\omega} \\
&= \frac{1}{2} \quad \text{because } \sin(4\pi + 2k) = \sin 2k \text{ for any value of } k
\end{aligned}$$

Therefore:

$$\begin{aligned}
S_{\text{rms}}{}^2 &= \langle\mathbf{S}\cdot\mathbf{S}\rangle = A^2\langle\sin^2(\omega t + \alpha)\rangle + B^2\langle\sin^2(\omega t + \beta)\rangle + C^2\langle\sin^2(\omega t + \gamma)\rangle \\
&= \frac{1}{2}\left(A^2 + B^2 + C^2\right)
\end{aligned}$$

Also:

$$\hat{\mathbf{S}} = Ae^{j\alpha}\,\mathbf{i} + Be^{j\beta}\,\mathbf{j} + Ce^{j\gamma}\,\mathbf{k} \quad \text{so that} \quad \hat{\mathbf{S}}\cdot\hat{\mathbf{S}}^* = A^2 + B^2 + C^2$$

therefore

$$S_{\text{rms}} = \sqrt{\frac{1}{2}\hat{\mathbf{S}}\cdot\hat{\mathbf{S}}^*}$$

Questions

5 (a) Let $\mathbf{r}_1 = (5, 3, 1)$ and $\mathbf{r}_2 = (-2, 0, 1)$ then $\mathbf{r}_2 - \mathbf{r}_1 = (-7, -3, 0)$ then the moment of $\mathbf{F}_1$ about $(-2, 0, 1)$ is given as:

$$\mathbf{M}_1 = \mathbf{F}_1 \times (\mathbf{r}_2 - \mathbf{r}_1) = (4, 3, -1) \times (-7, -3, 0)$$

That is:

$$\mathbf{M}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 3 & -1 \\ -7 & -3 & 0 \end{vmatrix} = 3\,\mathbf{i} + 7\,\mathbf{j} + 9\,\mathbf{k}$$

Let $\mathbf{r}_3 = (-1, 0, 2)$ and $\mathbf{r}_2 = (-2, 0, 1)$ then $\mathbf{r}_2 - \mathbf{r}_1 = (-1, 0, -1)$, so the moment of F_2 about $(-2, 0, -1)$ is given as:

$$\mathbf{M}_2 = \mathbf{F}_2 \times (\mathbf{r}_2 - \mathbf{r}_3) = (0, -6, 7) \times (-1, 0, -1)$$

That is:

$$\mathbf{M}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -6 & 7 \\ -1 & 0 & -1 \end{vmatrix} = 6\,\mathbf{i} - 7\,\mathbf{j} - 6\,\mathbf{k}$$

Total moment of both forces is then

$$\mathbf{M}_1 + \mathbf{M}_2 = 3\,\mathbf{i} + 7\,\mathbf{j} + 9\,\mathbf{k} + 6\,\mathbf{i} - 7\,\mathbf{j} - 6\,\mathbf{k} = 9\,\mathbf{i} + 3\,\mathbf{k}$$

(b) The magnitude of the moment is

$$|\mathbf{M}_1 + \mathbf{M}_2| = |9\,\mathbf{i} + 3\,\mathbf{k}| = \sqrt{(9^2 + 3^2)} = \sqrt{90} = 3\sqrt{10}$$

Questions Solutions