# Differentiation applications

**1** A metal window frame is to be constructed from a continuous strip of metal in the form of a rectangle surmounted by a semicircle. If the length of the metal strip is 10 m, find the dimensions of the window that permits the maximum amount of light to pass through.

Solutions Working

**2** One component of a whiskey still consists of a cylinder made from copper sheet with a cone of copper on the top, also made from copper sheet. To optimise production the still has to have the greatest volume possible but the gains made from production have to be offset by the high cost of pure copper sheet. For a given total volume and a circular base of a given radius prove that the amount of copper used is a minimum when the semi-vertical angle of the cone $\alpha$ is such that $\cos\alpha = 2/3$.

Working

**3** To create a visual impact an artist wishes to frame a rectangular picture in a rectangular frame where the sides of the frame are not parallel to the sides of the picture. The sides of a picture are $a$ and $b$ and the frame is constructed so that the each of the sides touches a corner of the picture. What is the maximum area of the frame?

Solutions Working

**4** An engineering component is in the form of solid sphere with a hole cut into it in the shape of a right circular cone. Show that the maximum possible volume of the conical hole is 8/27 times the volume of the sphere.

Solutions Working

**5** One component of a mechanical governor for an engine consists of metal sphere of radius $r$ through which a cylindrical hole has been cut. To optimise the efficiency of operation the cylindrical hole has to have the maximum surface area possible. Find the height $h$ of the cylinder when the surface area of the cylinder is a maximum.

Solutions Working

**6** The spectrum of black body radiation is given as:

$$u(\lambda) = \frac{8\pi c}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1}$$

where $u(\lambda)$ is the energy density between $\lambda$ and $\lambda + d\lambda$ where $\lambda$ is the wavelength of the radiation, $h$, $c$, $T$ are Plank's constant, the speed of light and the absolute temperature respectively and $k$ is the Boltzman constant. Show that for a given temperature the wavelength of maximum energy density $\lambda_{max}$ is given by the equation:

$$\lambda_{max} T = \frac{hc}{4{\cdot}965k}$$

*Hint*: Use the Newton-Raphson procedure to solve $\dfrac{x}{1 - e^{-x}} - 5 = 0$

Working

**7** In a submarine cable the range of signalling $r(x)$ is given by the expression:

$$r(x) = Ax^2 \ln(1/x)$$

where $A$ is a positive constant and $x$ is the ratio of the radius of the core to that of the cable.

(a) Find the stationary point of $r(x)$.
(b) Show that this stationary point maximises $r(x)$.

Solutions Working

**8** An electric cable has one end fixed to the floor and the other end fixed to a wall. Between these two points the cable hangs forming the shape given by:

$$y = 5e^{wx/f}$$

where $y$ is the height of a point on the cable distant $x$ horizontally from the lower fixed point and where $w$ and $f$ are constants.

(a) Find the change in the calculated value of $y$ produced by small errors in the measurement of$w$ and $f$.
(b) Show further that the percentage error in $y$ is equal to:

$$(wx/f) \cdot (\text{difference in percentage errors in } w \text{ and } f)$$

Solutions Working

**9** The deflection $y$ at the centre of a circular plate suspended at its circumference and uniformly loaded is given by:

$$y = \frac{kwd^4}{t^3}$$

where $w =$ total load, $d =$ diameter of the plate, $t =$ thickness of the plate and $k$ is a constant. If errors of $\pm 1\%$, $\pm 2\%$ and $\pm 1\%$ can occur in measuring the values of $w$, $d$ and $t$ respectively, use the first-order approximation to determine that maximum error possible in the calculated value of $y$.

Solutions Working

**10** In a tangent galvanometer the current is proportional to the tangent of the deflection of the needle. Show that if an observer always makes the same error when reading a deflection of the needle the percentage error in the current is least when the needle reading is $45^\circ$.

Working

**11** As part of a costing exercise it was necessary to know how costs changed when manufacturing processes deviated from their intended mode of operation. One process involved the construction of a right circular cone and it was required to know how the surface area $S$ varied with variations in the semi-vertical angle $\alpha$ given that the area of the slanting surface of a right circular cone with semi-vertical angle $\alpha$ is given as:

$$S = \pi h^2 \tan\alpha \sec\alpha$$

where $h$ is the height of the cone. Show that a small increase in the semi-vertical angle $\delta\alpha$ results in an increase in the surface area of the cone of:

$$\delta S = \frac{(1 + \sin^2\alpha)\delta\alpha}{\sin\alpha\cos\alpha}$$

and that this is least for a given value of $\delta\alpha$ when $\sin\alpha = 1/\sqrt{3}$.

Working

## Solutions

**1** 2·80 metres wide by 2·80 metres high

Questions Working

**3** $\dfrac{(a+b)^2}{2}$

Questions Working

**4** Volume of conical hole is $\frac{1}{3}\pi r^3(32/27)$

Questions Working

**5** $h = \sqrt{2r}$

Questions Working

**7** (a) $x = 1/\sqrt{e}$

Questions Working

**8** (a) $\frac{wxy}{f}\left(\frac{\delta w}{w} - \frac{\delta f}{f}\right)$

Questions Working

**9** $\pm 12\%$

Questions Working

## Working

**1** The maximum amount of light will pass when the window has maximum area. Keeping the perimeter of the window constant any change in the height of the window will be accompanied by a corresponding change in the width. Let $D$ be the diameter of the semicircle and $h$ the height of the rectangle. The perimeter of the window is then:

$$D + 2h + \frac{\pi D}{2} = 10$$

From this we find that:

$$h = 5 - \frac{D}{2}\left(1 + \frac{\pi}{2}\right)$$

The area of the window is then the area of the rectangle plus the area of the semicircle:

$$A = Dh + \frac{\pi}{2}\left(\frac{D}{2}\right)^2 = 5D - \frac{D^2}{2}\left(1 + \frac{\pi}{2}\right) + \frac{\pi D^2}{8} = 5D - \frac{D^2}{2}\left(1 + \frac{\pi}{4}\right)$$

Differentiating with respect to $D$ we find

$$\frac{dA}{dD} = 5 - D\left(1 + \frac{\pi}{4}\right) \quad \text{and} \quad \frac{d^2A}{dD^2} = -\left(1 + \frac{\pi}{4}\right) < 0$$

So the window has a maximum area when $\dfrac{dA}{dD} = 0$. That is when:

$$5 - D\left(1 + \frac{\pi}{4}\right) = 0 \quad \text{that is when} \quad D = 5\Big/\left(1 + \frac{\pi}{4}\right) = 2.80 \text{ metres to 2 dp.}$$

The height of the rectangular part of the window is then:

$$h = 5 - \frac{2{\cdot}80}{2}\left(1 + \frac{\pi}{2}\right) = 1{\cdot}40 \text{ metres to 2 dp}$$

giving the total height of the window as 2·80 metres to 2 dp.

Questions Solutions

**2** Let $H$ be the height of the cylinder below the conical top, let $h$ be the height of the cone and let $r$ be the radius of the cylinder and base of the cone. The volume of the cylinder is then $\pi r^2 H$ and the volume of the cone is $\pi r^2 h/3$. Let $V$ denote the volume of the still and we see that:

$$V = \pi r^2\left(H + \frac{h}{3}\right) \quad \text{so that} \quad H = \frac{V}{\pi r^2} - \frac{h}{3}$$

The surface area of the cylinder is given as the area of the base plus the area of the side which is:

$$S_{\text{cyl}} = \pi r^2 + 2\pi r H$$

The surface area of the conical cap is:

$$S_{\text{con}} = \pi r\sqrt{r^2 + h^2}$$

To find the minimum surface area for a fixed volume and fixed radius we must minimise:

$$S = \pi r^2 + 2\pi r H + \pi r\sqrt{r^2 + h^2} = \pi r^2 + 2\pi r\left(\frac{V}{\pi r^2} - \frac{h}{3}\right) + \pi r\sqrt{r^2 + h^2}$$

which we do by varying the height of the cone (which is accompanied by a corresponding change in the height of the cylinder). Now:

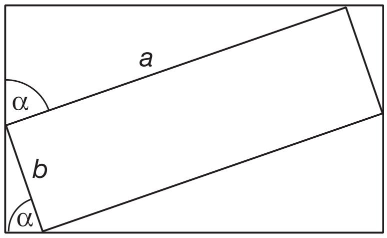
$$\frac{dS}{dh} = -\frac{2\pi r}{3} + \frac{\pi r h}{\sqrt{r^2 + h^2}} = 0 \quad \text{when} \quad \frac{h}{\sqrt{r^2 + h^2}} = \frac{2}{3}$$

that is when the semi-vertical angle $\alpha$ of the cone is such that $\cos\alpha = 2/3$. The fact that this is a minimum can be seen from the second derivative:

$$\frac{d^2S}{dh^2} = \frac{\pi r h}{\sqrt{(r^2 + h^2)}} = \frac{\pi r\left(r^2 + h^2\right)^{1/2} - \pi r h^2\left(r^2 + h^2\right)^{-1/2}}{(r^2 + h^2)} = \frac{\pi r^3}{(r^2 + h^2)^{3/2}} > 0$$

Questions Solutions

**3**




Each of the sides of the exterior rectangle is inclined to the the adjacent side of the interior rectangle by the same angle. If this angle is denoted by $\alpha$ then the side lengths of the exterior rectangle are:

$$a\sin\alpha + b\cos\alpha \quad \text{and} \quad a\cos\alpha + b\sin\alpha.$$

Consequently, the area of the exterior rectangle is given as:

$$\begin{aligned} A(\alpha) &= (a\sin\alpha + b\cos\alpha)(a\cos\alpha + b\sin\alpha) \\ &= ab\sin^2\alpha + ab\cos^2\alpha + \left(a^2+b^2\right)\sin\alpha\cos\alpha \\ &= ab + \left(a^2+b^2\right)\frac{\sin 2\alpha}{2} \end{aligned}$$

Therefore

$$\begin{aligned} A'(\alpha) &= \left(a^2+b^2\right)\cos 2\alpha \\ &= 0 \quad \text{when } 2\alpha = \pi/2 \text{ so that } \alpha = \pi/4 \end{aligned}$$

So there is a stationary point when $\alpha = \pi/4$. Since

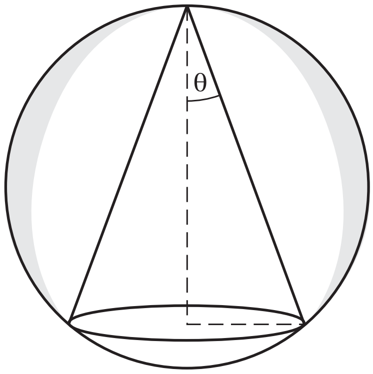
$$A''(\alpha) = -\left(a^2+b^2\right)2\sin 2\alpha < 0 \quad \text{when } \alpha = \pi/4$$

the stationary point is a maximum. The area of the exterior rectangle is then:

$$\begin{aligned} A(\alpha) &= ab + \left(a^2+b^2\right)\frac{\sin\pi/2}{2} \\ &= ab + \left(\frac{a^2+b^2}{2}\right) \\ &= \frac{(a+b)^2}{2} \end{aligned}$$

Questions Solutions

4




If the semi-vertical angle of the cone is $\theta$ then the height of the cone is $h = r + r\cos 2\theta$ and the volume of the cone is given as:

$$V = \frac{1}{3}\pi R^2 h \quad \text{where } R = r\sin 2\theta \text{ is the radius of the base.}$$

Therefore:

$$\begin{aligned}V(\theta) &= \frac{1}{3}\pi R^2 h \\ &= \frac{1}{3}\pi r^2 \sin^2 2\theta . r(1+\cos 2\theta) \\ &= \frac{1}{3}\pi r^3 \sin^2 2\theta + \frac{1}{3}\pi r^3 \sin^2 2\theta \cos 2\theta\end{aligned}$$

The volume $V(\theta)$ has a stationary point when $V'(\theta) = 0$ which, if $V''(\theta) < 0$ at that point, is a maximum. Now:

$$\begin{aligned}V'(\theta) &= \frac{1}{3}\pi r^3(2\sin 2\theta . 2\cos 2\theta) + \frac{1}{3}\pi r^3(2\sin 2\theta . 2\cos^2 2\theta - \sin^2 2\theta . 2\sin 2\theta) \\ &= \frac{1}{3}\pi r^3 2\sin 2\theta(2\cos 2\theta) + \frac{1}{3}\pi r^3 2\sin 2\theta(2\cos^2 2\theta - \sin^2 2\theta) \\ &= \frac{1}{3}\pi r^3 2\sin 2\theta\left[(2\cos 2\theta) + \left(2\cos^2 2\theta - [1-\cos^2 2\theta]\right)\right] \\ &= \frac{1}{3}\pi r^3 2\sin 2\theta\left(3\cos^2 2\theta + 2\cos 2\theta - 1\right) = 0 \quad \text{when}\end{aligned}$$

$\sin 2\theta = 0$ or $3\cos^2 2\theta + 2\cos 2\theta - 1 = 0$. That is when $\theta = 0$ or when $(3\cos 2\theta - 1)(\cos 2\theta + 1) = 0$, that is $\cos 2\theta = -1$ so $\theta = \pi/2$ or $3\cos 2\theta = 1$ whence $\cos 2\theta = 1/3$ and $\sin 2\theta = 2\sqrt{2}/3$.

Stationary points exist at $\theta = 0$ which is clearly a cone of zero volume, at $\theta = \pi/2$ which, again, is clearly a cone of zero volume and at $2\theta = \cos^{-1}(1/3)$ which must be the cone of maximum volume.

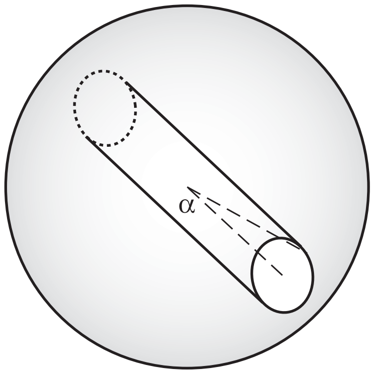
Now, when $\cos 2\theta = 1/3$:

$$V(\theta) = \frac{1}{3}\pi r^3 \sin^2 2\theta(1 + \cos 2\theta)$$
$$= \frac{1}{3}\pi r^3(8/9)(1 + 1/3) = \frac{1}{3}\pi r^3(32/27)$$

Whereas, the volume of the sphere is $\frac{4}{3}\pi r^3$ making the volume of the cone of maximum volume 8/27ths the volume of the sphere.

Questions Solutions

**5**




Let $\alpha$ be the angle between the radius of the sphere that has its end point on the edge of the cylindrical hole and radius along the axis of the cylinder then:

$$\frac{h/2}{r} = \cos\alpha \quad \text{so that } h = 2r\cos\alpha$$

Furthermore, the surface area of the cylinder, $S(\alpha)$, is given as:

perimeter of the circular end of the cylinder $\times$ height of the cylinder
$= 2\pi r \sin\alpha \times 2r\cos\alpha$
$= 2\pi r^2 \sin 2\alpha$

So that:

$$S'(\alpha) = 4\pi r^2 \cos 2\alpha \quad \text{and}$$
$$S''(\alpha) = -8\pi r^2 \sin 2\alpha$$

Therefore $S'(\alpha) = 0$ when $\cos 2\alpha = 0$, that is when $2\alpha = \pi/2$ so $\alpha = \pi/4$ and $S''(\pi/4) < 0$ which tells us that when $\alpha = \pi/4$ the surface area of the cylinder is a maximum. Also when $\alpha = \pi/4$ then

$$h = 2r\cos\pi/4 = \frac{2}{\sqrt{2}}r = \sqrt{2}r$$

Questions Solutions

**6**

$$u(\lambda) = \frac{8\pi c}{\lambda^5}\frac{1}{e^{hc/\lambda kT} - 1} \quad \text{therefore:}$$

$$\begin{aligned}
u'(\lambda) &= -5\frac{8\pi c}{\lambda^6}\frac{1}{e^{hc/\lambda kT} - 1} - \frac{8\pi c}{\lambda^5}\frac{(-hc/\lambda^2 kT)e^{hc/\lambda kT}}{(e^{hc/\lambda kT} - 1)^2} \\
&= -5\frac{8\pi c}{\lambda^6}\frac{1}{e^{hc/\lambda kT} - 1} + \frac{8\pi c}{\lambda^5}\frac{(hc/\lambda^2 kT)e^{hc/\lambda kT}}{(e^{hc/\lambda kT} - 1)^2} \\
&= -5\frac{8\pi c}{\lambda^6}\frac{1}{e^x - 1} + \frac{8\pi c}{\lambda^6}\frac{(x)e^x}{(e^x - 1)^2} \quad \text{where } x = hc/\lambda kT \\
&= \frac{8\pi c}{\lambda^6}\frac{1}{e^x - 1}\left(\frac{xe^x}{e^x - 1} - 5\right) \\
&= \frac{8\pi c}{\lambda^6}\frac{1}{e^x - 1}\left(\frac{x}{1 - e^{-x}} - 5\right) \\
&= 0
\end{aligned}$$

when

$$\frac{x}{1 - e^{-x}} - 5 = 0$$

A plot of $f(x) = \dfrac{x}{1 - e^{-x}} - 5$ will show that $x = 5$ is a good approximate solution to the equation $f(x) = 0$ [notice that $x = 0$ is not in the domain of the function].

Using the Newton-Raphson procedure we find that:

$$f(x) = \frac{x}{1 - e^{-x}} - 5 \quad \text{and so} \quad f'(x) = \frac{1(1 - e^{-x}) - xe^{-x}}{(1 - e^{-x})^2} \quad \text{where}$$

$$x_{n+1} = \frac{f(x_n)}{f'(x_n)} \quad \text{and} \quad x_0 = 5.$$

This can be programmed into a spreadsheet to produce the following result:

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ |
|---|---|---|---|
| 0 | 5.00 | 0.033918 | 0.972635 |
| 1 | 4.965127 | 1.28E-05 | 0.971896 |
| 2 | 4.965114 | 1.88E-12 | 0.971895 |
| 3 | 4.965114 | 0 | 0.971895 |

That is, when $\lambda = \lambda_{\max}$ then $x = hc/\lambda_{\max}kT = 4{\cdot}965$ to 3 dp. Hence:

$$\lambda_{\max}T = \frac{hc}{4{\cdot}965k}$$

Questions Solutions

**7** (a) Given $r(x) = Ax^2 \ln(1/x)$ then a stationary point exists where $r'(x) = 0$. Now:

$$r(x) = Ax^2 \ln(1/x) = -Ax^2 \ln x \text{ and so}$$
$$r'(x) = -2Ax \ln x - Ax$$

so stationary points exist where

$$r'(x) = -2Ax \ln x - Ax = -Ax(2\ln x + 1) = 0$$

That is when:

$$2\ln x = -1 \text{ that is when } x = 1/\sqrt{e}$$

(the case $x = 0$ is not permitted because $\ln 0$ is not defined)

(b) To determine the nature of the stationary point we need to consider the second derivative.

$$\begin{aligned} r''(x) &= -[Ax(2\ln x + 1)]' = -[A(2\ln x + 1) + Ax(2[1/x])] \\ \text{then } r''(1/\sqrt{e}) &= -\left[A\left(2\ln(1/\sqrt{e}) + 1\right) + A\left(1/\sqrt{e}\right)\left(2\left[1/\left(1/\sqrt{e}\right)\right]\right)\right] \\ &= -\left[A(-\ln(e) + 1) + A\left(1/\sqrt{e}\right)\left(2\sqrt{e}\right)\right] \\ &= -A\left(1/\sqrt{e}\right)\left(2\sqrt{e} + 1\right) \text{ since } \ln e = 1 \\ &< 0 \end{aligned}$$

Since $r''\left(1/\sqrt{e}\right) < 0$ the stationary point at $x = 1/e$ is a maximum.

Questions Solutions

**8** (a) Given $y = 5e^{wx/f}$ then:

$$\begin{aligned} \delta y &\cong \frac{\partial y}{\partial w}\delta w + \frac{\partial y}{\partial f}\delta f \\ &\cong 5\frac{x}{f}e^{wx/f}\delta w + 5wxe^{wx/f}\left(-1/f^2\right)\delta f \\ &\cong \frac{x}{f}y\delta w - \frac{wx}{f^2}y\delta f \\ &\cong \frac{wex}{f}y\frac{\delta w}{w} - \frac{wx}{f}y\frac{\delta f}{f} \\ &\cong \frac{wxy}{f}\left(\frac{\delta w}{w} - \frac{\delta f}{f}\right) \end{aligned}$$

(b) Therefore

$$\frac{\delta y}{y} \times 100 \cong \frac{wx}{f}\left(\frac{\delta w}{w} - \frac{\delta f}{f}\right) \times 100 \text{ as required}$$

Questions Solutions

**9** Given $y = \frac{kwd^4}{t^3}$ then the approximate error $\delta y$ obtained due to measurement errors of $\delta w$, $\delta d$ and $\delta t$ in $w$, $d$ and $t$ respectively is given as:

$$\begin{aligned}\delta y &\cong \frac{\partial y}{\partial w}\delta w + \frac{\partial y}{\partial d}\delta d + \frac{\partial y}{\partial t}\delta t \\ &= \frac{kd^4}{t^3}\delta w + 4\frac{kwd^3}{t^3}\delta d - 3\frac{kwd^4}{t^4}\delta t \\ &= \frac{y}{w}\delta w + 4\frac{y}{d}\delta d - 3\frac{y}{t}\delta t\end{aligned}$$

Therefore:

$$\begin{aligned}\frac{\delta y}{y} \times 100 &\cong \frac{\delta w}{w} \times 100 + 4\frac{\delta d}{d} \times 100 - 3\frac{\delta t}{t} \times 100 \\ &= \pm 1 + 4 \times (\pm 2) - 3 \times (\pm 1) \\ &= \pm 1 \pm 8 \pm 8 = \pm 12\%\end{aligned}$$

Questions Solutions

**10** The current $I(\theta)$ is proportional to the tangent of the deflection $\theta$ of the needle so that:

$$I(\theta) = k\tan\theta$$

The error $\delta I(\theta)$ in $I(\theta)$ due to a consistent error $\delta\theta$ in $\theta$ is then:

$$\begin{aligned}\delta I(\theta) &\cong \frac{\mathrm{d}I(\theta)}{\mathrm{d}\theta}\delta\theta \\ &= k\sec^2\theta\,\delta\theta\end{aligned}$$

Therefore:

$$\begin{aligned}\frac{\delta I(\theta)}{I(\theta)} &\cong \frac{k\sec^2\theta}{k\tan\theta}\delta\theta \\ &= \frac{2}{\sin 2\theta}\delta\theta\end{aligned}$$

This has a stationary value when $\left(\frac{2}{\sin 2\theta}\right)'$ that is when:

$$\left(\frac{2}{\sin 2\theta}\right) = \frac{-4\cos 2\theta}{\sin^2 2\theta} = 0$$

So when $\cos 2\theta = 0$, $2\theta = \pi/2$ so $\theta = \pi/4$ then the proportionate error in $I(\theta)$ has a stationary value.

Taking the second derivative:

$$\left(\frac{2}{\sin 2\theta}\right)'' = \left(\frac{-4\cos 2\theta}{\sin^2 2\theta}\right)' = \frac{8\sin 2\theta \times \sin^2 2\theta + 4\cos 2\theta \times 4\sin 2\theta\cos 2\theta}{\sin^4 2\theta}$$
$$> 0$$

when $\theta = \pi/4$.

Therefore the stationary point at $\theta = \pi/4$ is a minimum.

Questions Solutions

**11** Since $S(\alpha) = \pi h^2 \tan\alpha \sec\alpha$ then:

$$\begin{aligned} S'(\alpha) &= \pi h^2 (\sec^2\alpha\sec\alpha + \tan\alpha[\tan\alpha\sec\alpha]) \\ &= \pi h^2(\sec^3\alpha + \tan^2\alpha\sec\alpha) \\ &= \pi h^2 \sec\alpha(\sec^2\alpha + \tan^2\alpha) \\ &= \frac{S(\alpha)}{\tan\alpha}(\sec^2\alpha + \tan^2\alpha) \end{aligned}$$

The change $\delta S(\alpha)$ in $S(\alpha)$ due to a change in $\alpha$ of $\delta\alpha$ is given as:

$$\begin{aligned} \delta S(\alpha) &= S'(\alpha)\delta\alpha \\ &= \frac{S(\alpha)}{\tan\alpha}(\sec^2\alpha + \tan^2\alpha) \end{aligned}$$

Consequently the fractional change in $S(\alpha)$ is:

$$\frac{\delta S(\alpha)}{S(\alpha)} = \left(\frac{\sec^2\alpha + \tan^2\alpha}{\tan\alpha}\right)\delta\alpha = \left(\frac{1+\sin^2\alpha}{\sin\alpha\cos\alpha}\right)\delta\alpha$$

To find the minimum value of this we must differentiate $\dfrac{1+\sin^2\alpha}{\sin\alpha\cos\alpha}$ with respect to $\alpha$.

$$\left[\frac{1+\sin^2\alpha}{\sin\alpha\cos\alpha}\right]' = \left(\frac{2\sin\alpha\cos\alpha \times \sin\alpha\cos\alpha - [\cos^2\alpha - \sin^2\alpha][1+\sin^2\alpha]}{\sin^2\alpha\cos^2\alpha}\right)$$

$$= \left(\frac{2\sin^2\alpha\cos^2\alpha - [\cos^2\alpha - \sin^2\alpha] - [\cos^2\alpha\sin^2\alpha - \sin^4\alpha]}{\sin^2\alpha\cos^2\alpha}\right)$$

since $\cos^2\alpha = 1 - \sin^2\alpha$

$$= \left(\frac{2\sin^2\alpha - 2\sin^4\alpha - [1 - 2\sin^2\alpha] - [\cos^2\alpha\sin^2\alpha - \sin^4\alpha]}{\sin^2\alpha\cos^2\alpha}\right)$$

$$= \left(\frac{3\sin^2\alpha - 1}{\sin^2\alpha - \sin^4\alpha}\right) = 0 \text{ when } 3\sin^2\alpha = 1\text{, that is when } \sin\alpha = 1/\sqrt{3}$$

Questions Solutions