# Series

**1** The grand canonical partition function of a system of particles is given as:

$$Z = \prod_i Z_i$$

where, for bosons $Z_i = \sum_{n_i=0}^{\infty} e^{-\beta n_i(\varepsilon_i - \mu)}$, for fermions $Z_i = \sum_{n_i=0}^{1} e^{-\beta n_i(\varepsilon_i - \mu)}$ and for a Maxwell-Boltzmann gas $Z_i = \sum_{n_i=0}^{\infty} \frac{e^{-\beta n_i(\varepsilon_i - \mu)}}{n_i!}$.

(a) Show that for bosons $Z_i = \dfrac{1}{1 - e^{-\beta(\varepsilon_i - \mu)}}$

(b) Show that for fermions $Z_i = 1 + e^{-\beta(\varepsilon_i - \mu)}$

(c) Show that for a Maxwell-Boltzmann gas $Z_i = \exp\left(e^{-\beta(\varepsilon_i - \mu)}\right)$

## Working

**1** (a) Show that for bosons $Z_i = \dfrac{1}{1 - e^{-\beta(\varepsilon_i - \mu)}}$

$$\begin{aligned} Z_i &= \sum_{n_i=0}^{\infty} e^{-\beta n_i(\varepsilon_i - \mu)} \\ &= e^{-\beta 0(\varepsilon_i - \mu)} + e^{-\beta 1(\varepsilon_i - \mu)} + e^{-\beta 2(\varepsilon_i - \mu)} + \ldots + e^{-\beta N(\varepsilon_i - \mu)} + \ldots \end{aligned}$$

Let $x = e^{-\beta(\varepsilon_i - \mu)}$ so that $Z_i = 1 + x + x^2 + \ldots + x^N + \ldots = \dfrac{1}{1 - x}$. That is:

$$Z_i = \frac{1}{1 - e^{-\beta(\varepsilon_i - \mu)}}$$

(b) Show that for fermions $Z_i = 1 + e^{-\beta(\varepsilon_i - \mu)}$

$$\begin{aligned} Z_i &= \sum_{n_i=0}^{1} e^{-\beta n_i(\varepsilon_i-\mu)} \\ &= e^{-\beta 0(\varepsilon_i-\mu)} + e^{-\beta 1(\varepsilon_i-\mu)} \\ &= 1 + e^{-\beta(\varepsilon_i-\mu)} \end{aligned}$$

(c) Show that for a Maxwell-Boltzmann gas $Z_i = \exp\left(e^{-\beta(\varepsilon_i-\mu)}\right)$

$$\begin{aligned} Z_i &= \sum_{n_i=0}^{\infty} \frac{e^{-\beta n_i(\varepsilon_i-\mu)}}{n!} \\ &= \frac{e^{-\beta 0(\varepsilon_i-\mu)}}{0!} + \frac{e^{-\beta 1(\varepsilon_i-\mu)}}{1!} + \frac{e^{-\beta 2(\varepsilon_i-\mu)}}{2!} + \ldots + \frac{e^{-\beta N(\varepsilon_i-\mu)}}{N!} + \ldots \end{aligned}$$

Let $x = e^{-\beta(\varepsilon_i-\mu)}$ so that

$$Z_i = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^N}{N!} + \ldots = \exp(x)$$

That is:

$$Z_i = \exp\left(e^{-\beta(\varepsilon_i-\mu)}\right)$$

Question